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Kuban State University

COMPLEX ANALYSIS AND ITS APPLICATIONS

Materials of the International Conference,
Dedicated to the 70th anniversary of Corresponding Member
of the Russian Academy of Science V.N. Dubinin

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Российской академии наук В.Н. Дубинина

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The publication contains plenary lectures and reports at the International Conference *Complex Analysis and its Applications* in the following main areas of research: spaces of analytic functions, holomorphic mappings, multidimensional complex analysis, geometric function theory, quasiconformal mappings, potential theory, related problems of analysis.

Intended for scientists, postgraduates and undergraduates specializing in the field of complex, real and functional analysis.

The organization of the conference was supported by universities (Volgograd State University, Petrozavodsk State University) and academic institutes of the Russian Academy of Sciences (Steklov Mathematical Institute, M.V.Keldysh Institute of Applied Mathematics, Sobolev Institute of Mathematics SB RAS, Institute of Applied Mathematics FEB RAS.).

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Introduction

The International Conference "*Complex analysis and its applications*", dedicated to the 70th anniversary of Corresponding Member of the Russian Academy of Sciences, Prof. Vladimir Dubinin, was held from May 30 to June 5, 2021 at the Branch of the Kuban State University in Gelendzhik.

The purpose of this conference is to bring together mathematicians working in the area of complex analysis and its applications. At plenary lectures and invited talks in sections the participants discussed the current state and modern trends in this field. Plenary lectures were focused on modern methods of complex analysis and related fields and are of interest for both junior and senior mathematicians. Invited talks, presented at sections, cover a broad range of applications of various methods of complex analysis to problems in geometric function theory and approximation theory. Several recent results in quasiconformal mappings theory, potential theory, multidimensional complex analysis, functional analysis, theory of partial differential equations, and mathematical physics with applications also was discussed.

Working languages of the Conference: Russian, English.

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Vladimir Nikolaevich Dubinin
(to the 70th birthday)

Vladimir Nikolaevich Dubinin is a recognized specialist in complex analysis in the mathematical world, whose works in the field of geometric function theory have earned world fame.

V.N. Dubinin was born on June 2, 1951 in Vladivostok into a family of teachers. His mother, Nadezhda Evstafievna Dubinina, was a mathematics teacher and father, Nikolai Nikolaevich Dubinin, was a physics teacher, Honoured teacher of the USSR. In 1968 he finished with a silver medal boarding school №2 in Vladivostok (http://internat-n2.ru/). At his father recommendation, Volodya entered the Faculty of Mathematics of the Far Eastern State University. Why not Moscow? The explanation was simple: Georgy Konstantinovich Antonyuk worked there. Indeed, G.K. Antonyuk, a mathematics graduate of Leningrad State University, played an important role in the development of mathematics education in the Far Eastern capital, for many years leading the jury of the regional stage of the All-Russian Mathematical Olympiads for schoolchildren. He also taught a course in mathematical analysis for freshmen, including Volodya Dubinin. In 1970 G.K. Antonyuk moved to Krasnodar to the Kuban State University, which was granted its university status that year. In 1971 V.N. Dubinin was sent to Krasnodar to continue his studies. Professor I. Mityuk, Head of the Department of Theory of Functions, the first vice-rector for scientific work of the newly opened Kuban State University, becomes his scientific supervisor. The proposed by him theme for research, further fully developed, gave rise to the Kuban Mathematical School/ It turned out to be very prospective.. In a short time, the student V.N. Dubinin managed to obtain new scientific results. His graduation thesis "On a method of symmetrization and its application in cover theorems" was recognized as a laureate of the All-Russian competition of student works,and exhibited at the Exhibition of Achievements of the National Economy (the famous VDNKh). After graduating with honors from University in 1973, V. Dubinin entered the postgraduate course at the Department of Theory of Functions, Faculty of Mathematics, Kuban

State University. The creative atmosphere of the department and the regular school-conferences on the geometric theory of functions, conducted under the leadership of I.P. Mityuk, which alternated with colloquia on quasiconformal mappings organized by the Donetsk Institute of Applied Mathematics and Mechanics of the Academy of Sciences of the Ukrainian SSR promoted the development of Dubinin's talent. At the Academic Council of this institute in 1977, V. Dubinin defended his Ph.D. thesis on "Some symmetrization transformations and covering problems in the geometric theory of functions of a complex variable". Already by this time the contribution made by V. Dubinin in the development of symmetrization methods had been noticeable. Thanks to the new approach, he was able, in particular, to solve the Hayman problem on covering vertical segments, formulated in his monograph.

A deep understanding of the nature of these methods allowed V. Dubinin to find a different view of their capabilities and enrich the geometric theory of functions with new original methods of symmetrization, which made it possible to solve previously inaccessible problems. One of these tasks was the task of assessing the harmonic measure, put forward by A.A. Gonchar. Its solution was obtained thanks to the dissymmetrization transformation proposed by V. Dubinin. The polarization method, which goes back to the work of V. Volontis (1951) and was revived by V. Dubinin, obtained its new applications and sparkled with new colors. Dubinin's separating transformation of sets and condensers enriched the theory of symmetrization. It was his another method.

In 1989 V. Dubinin brilliantly defended his doctoral dissertation "The method of symmetrization in the geometric theory of functions" at the Institute of Mathematics of the Siberian Branch of the USSR Academy of Sciences (Novosibirsk). The first generalizing step in the presentation of developed symmetrization methods was the article by V. Dubinin "Symmetrization in the geometric theory of functions of a complex variable", published in 1994 in the journal "Uspekhi Matematicheskikh Nauk". In 2003 V. Dubinin published the textbook "Capacities of Capacitors in Geometric Function Theory". In the same year he was elected a Corresponding Member of the Russian Academy of Sciences (Far Eastern Branch). An important stage in the scientific biography of V. Dubinin became the publication in 2009 of his monograph "Capacities of capacitors and symmetrization in the geometric theory of functions of a complex variable". This fundamental work was published in English in 2013 and deservedly became the most frequently cited monograph on geometric function theory published in the last decade.

The original symmetrization methods proposed by V. Dubinin, made it possible to strengthen and develop many specific classical and modern results containing functional inequalities of an isoperimetric nature. In this direction, solutions are given to a number of open problems in the formulations of Polia, Szege, Fekete, Hayman, Jenkins, Goering, Gonchar, Vuorinen, Smale, and other mathematicians. Generalized condensers on the Riemann sphere with three or more plates were introduced and studied for the first time. A series of pioneering studies were carried out on the use of such capacitors in various branches of the geometric theory of functions. In particular, new metric properties of sets on the plane, inequalities for the Green's energy of a discrete charge were obtained; theorems on extremal decomposition with free poles, covering and distortion theorems for univalent and multivalent holomorphic functions are proved. V. Dubinin proposed a new version of circular symmetrization, which differs from the classical Polya symmetrization. It states that symmetrized sets and condensers are located on the Riemann surface of a function inverse to a Chebyshev polynomial of the first kind, or a suitable Zolotarev fraction. With the help of this symmetrization, a series of fundamentally new covering and distortion theorems is obtained for various classes of holomorphic functions, taking into account the critical values of the functions and the multiplicity of the covering. In particular, upper and lower bounds for the absolute values of critical values of normalized complex polynomials were found for the first time by symmetrization methods. In addition, new approaches to obtaining inequalities for polynomials and rational functions based on the use of univalent conformal mappings and majoration principles were proposed. As applications, generalizations and enhancements of the classical inequalities of Chebyshev, Markov, and Bernstein, as well as modern results of this kind, were established. Developed by Dubinin's general approach to obtaining geometric estimates for the Schwarzian derivative of holomorphic functions, based on the theory of capacitances of condensers and symmetrization, made it possible to establish the pioneering estimates of Schwarzian for univalent and multivalent functions both at the interior and at the boundary points of the domain of definition. Results and research methods of V. Dubinin have found applications in special questions of the theory of polynomials and rational functions, the theory of conformal mappings, potential theory, geometric function theory, the theory of partial differential equations and mathematical physics, in geometry, statistical physics and probability theory.

Since 1977 V. Dubinin has been working at the Far Eastern State (now Federal) University and since 1991, since the establishment of the Institute of Applied Mathematics of the Far Eastern Branch of the Russian Academy of Sciences, has been the Head of the laboratory of mathematical analysis of this institute. His students, who defended their Ph.D. theses, reached the level of independent research and are successfully work in our country and abroad, continuing the best traditions of the Russian mathematical school.

The versatility of talent, non-standard thinking and leadership qualities of Vladimir Nikolaevich are also recognized outside the boundaries of mathematical creativity. Participation in KVN of teachers and students, a lecture on complex analysis, read in the style of REP, poems and songs of his own composition - these and other facets of talent, wonderful human qualities of V. Dubinin never ceases to delight and amaze his colleagues and friends!

Plenary Lectures

On Uniform Convergence of Rational Approximants for Algebraic Functions[1]

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Approximations of the multi-valued functions by means of rational approximants have interesting features. For example, rational functions (being single-valued) can approximate only a branch of a multi-valued function and the domain of their convergence is a domain of holomorphicity for this particular branch. Moreover, the rational approximants themselves have singularities (i.e. poles of rational functions). Thus, they may model the singularities of the approximated function and therefore may converge in wider domains than polynomial approximations (i.e. their domains of convergence presents the holomorphic branch more globally).

A natural generalization of the Taylor polynomial approximants are the Padé rational approximants. Let f be a function holomorphic at infinity:

$$f(z) = \sum_{k=0}^{\infty} \frac{f_k}{z^k}. \tag{1}$$

A *diagonal Padé approximant* to f is a rational function $[n/n]_f = p_n/q_n$ of degree (n, n) that has maximal order of contact with f at infinity. Assume now that the germ (1) is analytically continuable along any path in $\mathbb{C} \setminus A$ for some fixed set A. Suppose further that this continuation is multi-valued in $\overline{\mathbb{C}} \setminus A$, i.e., f has branch-type singularities at some points in A. This class of functions is denoted by $f \in \mathcal{A}(\overline{\mathbb{C}} \setminus A)$.

In [1] Nuttall has conjectured that *for any function f in $\mathcal{A}(\overline{\mathbb{C}} \setminus A)$ with any finite number of branch points that are arbitrarily positioned in the complex plane,*

$$\sharp\, A < \infty \quad \text{and} \quad A \subset \mathbb{C},$$

and with an arbitrary type of branching singularities at those points, the diagonal Padé approximants converge to f in logarithmic capacity, i.e., $\forall\ \varepsilon > 0$,

$$\lim_{n\to\infty} cap\,(\{z \in K : \ |f(z) - [n/n]_f(z)| > \varepsilon\}) \,=\, 0\,, \quad K \subset \overline{\mathbb{C}} \setminus \Delta\,, \tag{2}$$

away from the system of cuts Δ of minimal logarithmic capacity, i.e.,

$$\mathrm{cap}(\Delta) = \min_{\partial D : D \in \mathcal{D}_f} \mathrm{cap}(\partial D)\,, \tag{3}$$

where by $\mathcal{D}_f$ is denoted the collection of all connected domains containing the point at infinity in which f is holomorphic and single-valued.

In [2 - 4] H. Stahl for a multi-valued $f \in \mathcal{A}(\overline{\mathbb{C}} \setminus A)$ with $\mathrm{cap}(A) \,=\, 0$ proved:

- the *existence* of a domain $D^* \in \mathcal{D}_f$ such that $\Delta = \partial D^*$satisfies (3);
- *weak (n-th root) asymptotics* for the denominators of the $[n/n]_f$:

$$\lim_{n\to\infty} \frac{1}{n} \log|q_n(z)| = \int \log|z-t|\, d\omega_\Delta(t) =: -V^{\omega_\Delta}(z), \qquad z \in D^*,$$

  where V^{ω_Δ} is the log potential of the equilibrium measure ω_Δ : $V^{\omega_\Delta} =$ const. a.e. on Δ;
- *convergence theorem* (2) conjectured by Nuttall.

[1]This work was supported by Russian Science Foundation (Grant RSF-19-71-30004).

We report here on some results of our joint project with Maxim Yatselev from Indiana University - Purdue University in Indianapolis and Moscow Center of Fundamental and Applied Mathematics.

In this lecture we focus on the strong (Bernshtein-Szego type) asymptotics of Padé approximants, providing their uniform convergence. In other words,we focus on identifying the limit

$$\lim_{n\to\infty}\frac{q_n}{\Phi^n}=?\quad\text{in}\quad D^*$$

of *the Nuttall-Stahl polynomials* q_n (denominators of the diagonal Padé approximants for $f\in\mathcal{A}(\overline{\mathbb{C}}\setminus A)$, $\sharp\, A<\infty$), where Φ is a properly chosen normalizing function.

We start with a statement on the strong asymptotics formulas (obtained in [5]) for the diagonal Padé approximants of analytic functions with finitely many branch points of the algebraic-logarithmic character, which are situated on the complex plane in a *generic position* (GP). As it turns out, the asymptotics of Padé approximants is described with the help of certain functions that solve a specific boundary value problem on a Riemann surface $\mathfrak{R}$ that corresponds to the minimal capacity contours. These functions (we denote them by S) are analogues of the classical Szego functions in the strong asymptotics of orthogonal polynomials. However, unlike the classical case the function S may have zeros on $\mathfrak{R}$ and the number of these zeros is related to the *genus* of $\mathfrak{R}$. Positions of these zeros (depending on n) are described by means of a certain Jacobi problem of the inversion of the Abelian integrals on $\mathfrak{R}$. Projections of these zeros from $\mathfrak{R}$ to $\mathbb{C}$ correspond to *spurious* or *false* poles and *extra interpolation* points for Padé approximants, that destroys their convergence. Thus, the control of these zeros of S is an important ingredient in proving the uniform convergence of the rational approximants.

Then in the second part of our lecture, we consider an application of the strong asymptotics (discussed above) for the rational Padé approximants of the algebraic functions to prove the famous conjectures about functional analogues of the celebrated Tue–Sigel–Rota Theorem on the lower bound of the rate of approximation of an algebraic number by means of rational numbers. It states: let α be an arbitrary algebraic number, then for all $\varepsilon>0\ \ \exists\, C(\varepsilon)$ (non effective), such that

$$\left|\alpha-\frac{p}{q}\right|\geqslant\frac{C(\varepsilon)}{q^{2+\varepsilon}},\quad\forall p/q\in\mathbb{Q}. \tag{4}$$

The algebraic numbers for which the Diophantine inequality (4) is valid with $\varepsilon=0$ are called *slowly approximated irrationalities* (SAI). It is not difficult to see, that for α to be (SAI) is equivalent to have bounded incomplete quotions (i.e. all the coefficient of its continued fraction expansion are bounded by some constant). Thus, quadratic irrationalities are (SAI), however to disprove (or prove) this fact even for $\sqrt[3]{2}$ is a difficult open problem. Introducing a non-Archimedean norm on the field of Laurent series at infinity, we can get a functional analog of Diophantine approximants, in which the convergents of functional continued fraction (i.e. diagonal Padé approximants) for f - power series (1) corresponds to continued fraction for number α. We denote

$$\nu_n(f)=\sup\{\nu(f-r):r\in\mathcal{R}_n\}=\nu(f-[n/n]_f),\qquad \nu(f(z)):=\underset{z=\infty}{\operatorname{ord\,zero}}\,f\,, \tag{5}$$

where $\mathcal{R}_n$ is a class of rational functions of order not greater than n. The functional analog of Tue-Sigel-Rota Theorem (or so called Kolchin conjecture, see [6, 7]) is stated as follows. Let f be algebraic function (or solution of differential equation with coefficients over the field of rational functions), then for any $\varepsilon>0\quad\exists C(f)$:

$$\nu_n(f)\ <\ (2+\varepsilon)n\ +\ C(f),\qquad n\in\mathbb{N}\,. \tag{6}$$

The Kolchin conjecture is proved: with "non-effective" constants $C(f)$ in [8] and with "effective" constants $C(f)$ in [9, 10]. However, A. Gonchar in [11], D. and G.

Chudnovskies in [10] and H. Stahl in [12] put forward a stronger version of Kolchin conjecture claiming that *for any algebraic function inequality* (6) *remains true even for* $\varepsilon = 0$ (!):

$$\nu_n(f) \, < \, 2\,n \, + \, C(f), \qquad n \in \mathbb{N}\,. \tag{7}$$

We have shown that the Gonchar - Chudnovskies - Stahl conjecture on $\varepsilon = 0$ holds true for any algebraic function f, whose Padé approximants possess the strong asymptotics as in [5], namely the branch points of f are in the generic position (i.e. the GP condition is fulfilled). Moreover, we have the effective constant

$$C(f) \,:=\, g \,+\, d\,, \qquad \text{where} \quad g := \mathrm{gen}(\mathfrak{R}), \quad d := \deg\left(\mathrm{Discrim}(f(z))\right)\,,$$

where $\mathfrak{R}$ is the Stahl's Riemann surface.

In the final part of the lecture, we discuss possibilities to avoid the extra conditions (like the GP condition) in the proof of strong asymptotics of Padé approximants. For this we recall the approach to prove the strong asymptotics in [5]. The approach is based on the matrix Riemann-Hilbert problem for 2x2 matrix analytic functions. Let $\mathcal{Y}$ be a 2×2 matrix function. Consider the following Riemann-Hilbert problem for $\mathcal{Y}$ (RHP$\mathcal{Y}$):

(a) $\mathcal{Y}$ is analytic in $\mathbb{C} \setminus \Delta$ and $\lim\limits_{z\to\infty} \mathcal{Y}(z) z^{-n\sigma_3} = \mathcal{I}$, where $\mathcal{I}$ is the identity matrix and σ_3 is the Pauli matrix $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$;

(b) $\mathcal{Y}$ has continuous traces on each analytic arc Δ_k constituting Δ that satisfy

$$\mathcal{Y}_+ = \mathcal{Y}_- \begin{pmatrix} 1 & \rho \\ 0 & 1 \end{pmatrix}$$

where $\rho(t) = (f^+ - f^-)(t), \quad t \in \Delta =: \bigcup \Delta_k$;

(c) $\mathcal{Y}$ has a prescribed asympytotics in the neighborhoods of the projections on $\mathbb{C}$ of the branch points of $\mathfrak{R}$.

The connection between RHP$\mathcal{Y}$ and polynomials orthogonal with respect to ρ (i.e.the denominators q_n of Padé approximants $[n/n]_f$) was first realized by Fokas, Its, and Kitaev [13, 14] and and is as follows. If a solution of RHP$\mathcal{Y}$ exists, then it is unique. Moreover, in this case $\deg(q_n) = n$, $R_{n-1}(z) \sim z^{-n}$ as $z \to \infty$, and the solution of RHP$\mathcal{Y}$ is given by

$$\mathcal{Y} = \begin{pmatrix} q_n & R_n \\ m_{n-1} q_{n-1} & m_{n-1} R_{n-1} \end{pmatrix}, \quad R_n(z) := q_n(z) f(z) - p_n(z) = \mathcal{O}\left(1/z^{n+1}\right) \tag{8}$$

where m_n is a constant such that $m_{n-1} R_{n-1}(z) = z^{-n}[1 + o(1)]$ near infinity.

We see from condition (a) that (RHP$\mathcal{Y}$) depends on parameter n. Thus, due to (8), to find asymptotics of q_n, R_n we have to find solution of (RHP$\mathcal{Y}$) for big n. A steepest decent method to find $\mathcal{Y}$ when $n \to \infty$ was proposed in [15] and developed in details by P. Deift, his collaborators and numerous successors (see the textbook [16]). One of the most important ingredient of the method is to find the local solution of (RHP$\mathcal{Y}$) in the neighborhoods of singularities of $\mathcal{Y}$. Regarding the asymptotics of Padé approximants (q_n, R_n) for analytic functions with finite set of branch points A, this special local analysis is required for the end points of analytic arcs Δ_k (we denote this set of point as E). We now specify (see [5]) that condition (GP) assumes that

(i) each point in $E \bigcap A$ is incident with exactly one arc from $\bigcup \Delta_k$;

(ii) each point in $E \setminus A$ is incident with exactly three arcs from $\bigcup \Delta_k$.

Under this condition the local analysis was done in [5]: for the case (i) the asymptotics are described by the Bessel function and for the case (ii) by means of the Airy functions (see also pioneering papers on the local asymptotics [18], [17]). The work on these additional local problems, needed to avoid condition (GP), is in progress. We shall briefly discuss these results in the conclusion of the lecture.

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Bilateral estimates of functionals of mathematical physics[1]

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Let $\Omega \subset \mathbb{C}$ be a domain, $\Omega \neq \mathbb{C}$. For $p \in [1,2]$ we study the inequality

$$\left(\iint_\Omega |u(z)|^p dxdy\right)^{2/p} \le \Lambda_{p-1}(\Omega) \iint_\Omega |\nabla u(z)|^2 dxdy \quad \forall u \in C_0^1(\Omega),$$

where $z = x + iy$, $\Lambda_{p-1}(\Omega) \in (0, \infty]$ is the infimum of admissible constants at this place. We generalize the known results from the papers [1] and [2] using

$$\rho(z,\Omega) = \mathrm{dist}(z, \partial\Omega) := \inf_{w \in \partial\Omega} |z - w|, z \in \Omega,$$

$$\rho(\Omega) := \sup_{z \in \Omega} \rho(z,\Omega), \quad I_p(\Omega) := \iint_\Omega \rho^{2p/(2-p)}(z,\Omega) dxdy.$$

Theorem 1. *Let $\Omega \subset \mathbb{C}$ be a finitely connected domain, $\Omega \neq \mathbb{C}$. Let $\mathbb{K}$ be a compact set such that $\mathbb{K} \subset \Omega$ and $\Omega \setminus \mathbb{K}$ is a domain, in particular, $\mathbb{K} = \emptyset$.*
Then one has that $\Lambda_1(\Omega \setminus \mathbb{K}) < \infty \iff \rho(\Omega \setminus \mathbb{K}) < \infty$ and for every $p \in [1,2)$

$$\Lambda_{p-1}(\Omega \setminus \mathbb{K}) < \infty \iff I_p(\Omega \setminus \mathbb{K}) < \infty.$$

Theorem 2. *Let $\Omega \subset \mathbb{C}$ be a domain such that $\Omega \neq \mathbb{C}$ and $L(z_0, r) \setminus \Omega \neq \emptyset$ for any circle of the kind $L(z_0, r) = \{z \in \mathbb{C} : |z - z_0| = r\}$ with $z_0 \in (\partial\Omega) \setminus \infty$ and $r > 0$.*
Then one has that

$$\rho^2(\Omega)/j_0^2 \le \Lambda_1(\Omega) \le \frac{\Gamma^{16}(1/4)}{16\pi^8} \rho^2(\Omega),$$

and for every $p \in [1,2)$

$$\frac{2-p}{4} \left(I_p(\Omega)\right)^{2/p-1} \le \Lambda_{p-1}(\Omega) \le \frac{\Gamma^{16}(1/4)}{16\pi^8} \left(I_p(\Omega)\right)^{2/p-1},$$

where Γ is Euler's gamma function, j_0 is the first zero of the Bessel function J_0.

In addition, we prove analogs of theorems 1 and 2 for plane domains with uniformly perfect boundaries as well as for n-dimensional spatial domains that are λ-close-to-convex. These results will be published in [3].

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REFINED MODEL OF LITHOSPHERIC PLATES IN THE PROBLEM TSUNAMI FORECAST

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In the works of the authors, taking into account the scale of the Earth and the size of its crust in the zones of land and ocean, semi-infinite Kirchhoff plates are traditionally accepted as lithospheric plates, which are located on a deformable base and have sufficiently reliably justified their use in various works. The ends of the lithospheric plates are parallel and opposite [1]. It is of interest to study the problem in the assumption of a more complex rheology of lithospheric plates, as well as a more complex shape of the lithospheric plate, for example, in the form of a rectangular wedge. In particular, a more complex model is linear elastic lithospheric plates described by systems of Lame differential equations. The block element method allows solutions of boundary value problems for systems of partial differential equations with constant coefficients, that is, vector ones, to be represented decomposed by solutions of scalar boundary value problems, that is, for individual equations.
This kind of representation is quite well known, it is a representation in the form potentials, vortex and potential for the Lame equations. Another approach leads to a similar decomposition as a result of applying the Galerkin transform.
However, until recently, the approaches were used only in the entire space, or in boundary value problems posed in classical domains. These include the half-space or layered regions, as well as the regions obtained as a result of constructing various groups of transformations of the space. This could not be done due to the complexity of satisfying the boundary conditions when the boundary value problem it is placed in a non-classical area, for example, in a wedge. The block element method allowed us to overcome this complexity, that is, to obtain representations of solutions to vector boundary value problems using scalar ones. The authors have developed approaches for solving these boundary value problems. They are described in [2,3]. However, in the process of constructing scalar boundary value problems for lithospheric plates, which are obtained in connection with the transformation of more complex vector boundary value problems, problems may arise due to the cumbersomeness of individual equations obtained as combinations of Helmholtz equations.

1. As an example, we consider the boundary value problem for the Lame equation in the domain Ω, in the first quadrant of a rectangular Cartesian coordinate system, under certain non-zero time-harmonic boundary conditions

$$\begin{aligned} &L_{mn}(u_n) = 0, \quad L_{mn}(u_n) = \delta_{mn}\Delta + \sigma\partial_m^2\partial_n^2 - p, \quad m,n = 1,2,3, \\ &\sigma = \mu^{-1}(\lambda+\mu), \quad \partial_m^h = \frac{\partial^h}{\partial x_m^h} \end{aligned} \tag{1}$$

δ_{mn} - the Kronecker symbol, Δ - Laplacian $\Delta = \partial_1^2 + \partial_2^2$,
We implement the Galerkin transformation by putting [4]

$$\mathrm{u}_1 = \begin{vmatrix} \chi_1 & \mathrm{L}_{12} & \mathrm{L}_{13} \\ \chi_2 & \mathrm{L}_{22} & \mathrm{L}_{23} \\ \chi_3 & \mathrm{L}_{32} & \mathrm{L}_{33} \end{vmatrix}, \quad \mathrm{u}_2 = \begin{vmatrix} \mathrm{L}_{11} & \chi_1 & \mathrm{L}_{13} \\ \mathrm{L}_{21} & \chi_2 & \mathrm{L}_{23} \\ \mathrm{L}_{21} & \chi_3 & \mathrm{L}_{33} \end{vmatrix}, \quad \mathrm{u}_3 = \begin{vmatrix} \mathrm{L}_{11} & \mathrm{L}_{12} & \chi_1 \\ \mathrm{L}_{21} & \mathrm{L}_{22} & \chi_2 \\ \mathrm{L}_{21} & \mathrm{L}_{32} & \chi_2 \end{vmatrix}$$

and denoting $\mathrm{T_i} = \Delta\chi_\mathrm{i}$. Then the system of equations (1) is reduced to a biharmonic equation with respect to the Galerkin functions $\mathrm{T_i}$ of the form

$$(\Delta\Delta - \mathrm{k}^2)\mathrm{T_i} = 0, \quad \mathrm{k} = \mathrm{const}$$

In the following, we consider a two-dimensional scalar boundary value problem for a biharmonic equation in the domain of the first quadrant when we set the harmonic functions and the first normal derivatives to the boundaries at the boundary

$$\begin{aligned}
&\mathrm{Lu} =(\partial_1^4 + 2\partial_1^2\partial_2^2 + \partial_2^4 - \mathrm{k}^2)\mathrm{u} = 0, \quad \mathrm{k}^2= \rho \mathrm{h}^{-2}\omega^2 12(1-\nu^2)\mathrm{E}^{-1} \\
&\mathrm{u}_1(\mathrm{x}_1, 0) = \mathrm{g}_1(\mathrm{x}_1, 0), \quad \partial_2^1 \mathrm{u}(\mathrm{x}_1, 0) = \mathrm{b}_1(\mathrm{x}_1, 0) \\
&\mathrm{u}_2(0, \mathrm{x}_2) = \mathrm{g}_2(0, \mathrm{x}_2), \quad \partial_1^1 \mathrm{u}_2(0, \mathrm{x}_2) = \mathrm{b}_2(0, \mathrm{x}_2)
\end{aligned} \tag{2}$$

We introduce the operator of the boundary problem, taking, that Ω is, the first quadrant, as the domain of its definition, and for external influences, the time-harmonic vertical displacements of the boundaries $\mathrm{u}e^{-i\omega t}$ and the same angles of rotation at the boundary. Here - ρ the linear density of the material, h - the thickness of the plate, ω - the frequency of harmonic influences on the plate, ν and E - the Poisson's ratio and Young's modulus of the plate material, respectively.

2. The block element method can be applied directly to the equation of the boundary value problem (2). Using the algorithm of the block element method, which includes the steps of external algebra, external analysis, and factor topology. In the case of a problem of the second kind, the functions and derivatives at $x_1 = 0$ and at $x_2 = 0$ are given, viz.

$$\mathrm{u}(0, \mathrm{x}_2), \quad \mathrm{u}(\mathrm{x}_1, 0), \quad \partial_1 \mathrm{u}(0, \mathrm{x}_2), \quad \partial_2 \mathrm{u}(\mathrm{x}_1, 0)$$

Then we can implement the algorithm stage of the block element method called "external form". It leads to the immersion of the boundary value problem in the topological space and to the further construction of the external form and the functional equation. In the case of the scalar problem under consideration, a unique functional equation of the form is obtained

$$\begin{aligned}
&\left[({\alpha_1}^2{+}{\alpha_2}^2)^2{-}\mathrm{k}^2\right]\mathrm{U}(\alpha_1, \alpha_2) = \omega(\alpha_1, \alpha_2) \\
&\omega(\alpha_1, \alpha_2) = \mathrm{i}\alpha_2 \mathrm{T}_1(\alpha_1, 0) + \mathrm{i}\alpha_1 \mathrm{T}_2(0, \alpha_2) - \mathrm{S}_1(\alpha_1, 0) - \mathrm{S}_2(0, \alpha_2) - \mathrm{P}_1(\alpha_1, 0) - \mathrm{P}_2(0, \alpha_2) \\
&\mathrm{P}(\alpha_1, \alpha_2) = F(\alpha_1, \alpha_2)\mathrm{p}(\mathrm{x}_1, \mathrm{x}_2), \quad \mathrm{S}_\mathrm{n} = F(\alpha_1, \alpha_2)\mathrm{s}_\mathrm{n}, \quad \mathrm{T}_\mathrm{n}{=}F(\alpha_1, \alpha_2)\mathrm{t}_\mathrm{n}
\end{aligned} \tag{3}$$

Here, the external form after applying the Fourier transform, and the notation system are introduced

$$\begin{aligned}
&\partial_2^3 \mathrm{U}(\alpha_1, 0) + \partial_1^2\partial_2^1 \mathrm{U}(\alpha_1, 0) = \mathrm{S}_1(\alpha_1, 0), \quad \partial_2^2 \mathrm{U}(\alpha_1, 0) + \partial_1^2 \mathrm{U}(\alpha_1, 0) = \mathrm{T}_1(\alpha_1, 0) \\
&\partial_1^3 \mathrm{U}(0, \alpha_2) + \partial_1^1\partial_2^2 \mathrm{U}(0, \alpha_2) = \mathrm{S}_2(0, \alpha_2), \quad \partial_2^2 \mathrm{U}(0, \alpha_2) + \partial_1^2 \mathrm{U}(0, \alpha_2) = \mathrm{T}_2(0, \mathrm{x}_2) \\
&[\,(-\mathrm{i}\alpha_2)^2\partial_2^1 + (-\mathrm{i}\alpha_1)^2\partial_1^1\,]\,\mathrm{U}(\alpha_1, 0) + [\,(-\mathrm{i}\alpha_2)^3{+}(-\mathrm{i}\alpha_1)^2(-\mathrm{i}\alpha_2)\,]\,\mathrm{U}(\alpha_1, 0) = \mathrm{P}_1(\alpha_1, 0) \\
&[\,(-\mathrm{i}\alpha_2)^2\partial_2^1 + (-\mathrm{i}\alpha_1)^2\partial_1^1\,]\,\mathrm{U}(0, \alpha_2) + [\,(-\mathrm{i}\alpha_1)^3{+}(-\mathrm{i}\alpha_2)^2(-\mathrm{i}\alpha_1)\,]\,\mathrm{U}(0, \alpha_2) = \mathrm{P}_2(0, \alpha_2)
\end{aligned}$$

$F(\alpha_1, \alpha_2)$-the Fourier transform operator, - its parameters. The next step is to perform an external analysis, including factorization of the coefficient of the functional equation, calculation of the Lehrer residue forms, construction of pseudo-differential equations and their solutions. Solutions of pseudodifferential equations are introduced in external forms and allow us to obtain a representation of the solution of the boundary value problem in the form of a packed block element.
The general representation of the solution of the boundary value problem, taking into account (3), has the form

$$\mathrm{u}(\mathrm{x}_1, \mathrm{x}_2) = \frac{1}{4\pi^2} \iint\limits_{\mathbb{R}^2} \frac{\omega(\alpha_1, \alpha_2)}{[\,({\alpha_1}^2{+}{\alpha_2}^2)^2{-}\mathrm{k}^2\,]} \mathrm{e}^{-\mathrm{i}(\alpha_1 \mathrm{x}_1 + \alpha_2 \mathrm{x}_2)} \mathrm{d}\alpha_1 \mathrm{d}\alpha_2$$

$\mathbb{R}^2$ - two-dimensional space of real numbers. This approach allows us to build a solution representation for complicated models of lithospheric plates.
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Approximation by simple partial fractions in unbounded domains

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We consider uniform approximation by simple partial fractions (logarithmic derivatives of polynomials)

$$r(z) = \sum_{k=1}^{n} \frac{1}{z - a_k}, \qquad a_k \in \mathbb{C},$$

inside a domain $D \subset \mathbb{C}$ provided that the poles a_k of approximating fractions are taken in the boundary ∂D.

Let $SF(E)$ denote the family of all simple partial fractions having poles in a set $E \subset \mathbb{C}$; $A(D)$ denotes the space of functions holomorphic in a domain D equipped with the topology of uniform convergence on compact subsets of D.

Korevaar [1] proved that $SF(\partial D)$ is dense in $A(D)$ for any bounded simply connected domain D. For unbounded domains, a nice result was proved by Elkins [2]: if a simply connected domain D lies in a half-plane and does not contain any half-plane, then $SF(\partial D)$ is dense in $A(D)$.

We present the following result [3].

Theorem. *Let a domain $D \subset \mathbb{C}$ have a boundary consisting of Jordan curves*

$$\xi_k = \{\xi_k(t) : t \in [0;1]\} \subset \overline{\mathbb{C}}, \qquad k = 1, \dots, m,$$

with the following properties:

(1) *each ξ_k is rectifiable in $\overline{\mathbb{C}}$; different ξ_k and ξ_j intersect only at ∞;*

(2) *for every $k = 1, \dots, m$, we have*

$$\xi_k(0) = \xi_k(1) = \infty; \qquad \xi_k(t) \in \mathbb{C} \text{ for } 0 < t < 1,$$

$$\lim_{t\to 0} \arg \xi_k(t) = \alpha_k, \qquad \lim_{t\to 1} \arg \xi_k(t) = \beta_k,$$

where $0 \le \alpha_1 \le \beta_1 \le \alpha_2 \le \beta_2 \le \dots \le \alpha_m \le \beta_m \le 2\pi$, and besides $\arg \xi_k(t)$ has bounded variation near $t = 0$ and $t = 1$.

If for any positive integer n the set

$$V^n := \bigcup_{k=1}^{m} \{z^n : \alpha_k \le \arg z \le \beta_k\}$$

does not belong to a closed half-plane, then $SF(\partial D)$ is dense in $A(D)$.

Conversely, if V^n lies in an open half-plane for some n, then $SF(\partial D)$ is not dense in $A(D)$.

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Multipoint analogues of the Carathéodory and Schur criteria [1]

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Напомним, что функция $f(z)$, голоморфная в круге $\mathbb{D} = \{|z| < 1\}$, называется функцией Каратеодори, если $\Re f(z) \geq 0$, и называется функцией Шура, если $|f(z)| \leq 1$. Множества функций Каратеодори и функций Шура будем обозначать соответственно через $\mathcal{B}^{\mathsf{c}}$ и $\mathcal{B}^{\mathsf{s}}$. В множествах $\mathcal{B}^{\mathsf{c}}$ и $\mathcal{B}^{\mathsf{s}}$ обычно выделяют следующие непересекающиеся подмножества:

$\mathcal{B}^{\zeta}_0$ – множество постоянных функций вида $f(z) \equiv \lambda$, где $\Re\lambda = 0$, если $\zeta = \mathsf{c}$, и $|\lambda| = 1$, если $\zeta = \mathsf{s}$,

$\mathcal{B}^{\mathsf{c}}_N$ $(N \in \mathbb{N})$ – множество рациональных функций вида $f(z) = \lambda_0 + \sum_{k=1}^{N} \lambda_k \frac{t_k - z}{t_k + z}$, где $t_1, \ldots, t_N$ попарно различны, $|t_k| = 1$, $\Re\lambda_0 = 0$, $\lambda_k > 0$, $k = 1, \ldots, N$,

$\mathcal{B}^{\mathsf{s}}_N$ $(N \in \mathbb{N})$ – множество рациональных функций вида $f(z) = \gamma \prod_{k=1}^{N} \frac{z - e_k}{1 - z\overline{e}_k}$, где $|\gamma| = 1$, $e_k \in \mathbb{D}$, $\overline{e}_k$ – комплексно сопряженное к e_k, $k = 1, \ldots, N$,

$\mathcal{B}^{\zeta}_{\infty} := \mathcal{B}^{\zeta} \setminus (\cup_{N \in \mathbb{Z}_+} \mathcal{B}^{\zeta}_N)$, $\zeta = \mathsf{c}, \mathsf{s}$.

В 1907 году Каратеодори в [1] и в 1918 году Шура в [2] нашли необходимые и достаточные условия, при которых заданный формальный степенной ряд является рядом Тейлора функции Каратеодори или функции Шура. Для краткости сформулируем критерии Каратеодори и Шура в виде единого критерия, в котором случай $\zeta = \mathsf{c}$ совпадает с критерием Каратеодори, а случай $\zeta = \mathsf{s}$ совпадает с критерием Шура.

Критерий Каратеодори–Шура. *Пусть $f(z) = \sum_{k=0}^{\infty} a_k z^k$ – формальный степенной ряд,*

$$A^f_n := \begin{pmatrix} a_0 & a_1 & \ldots & a_{n-1} \\ 0 & a_0 & \ldots & a_{n-2} \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & a_0 \end{pmatrix}, \quad \tilde{A}^f_n := \begin{pmatrix} \overline{a}_0 & 0 & \ldots & 0 \\ \overline{a}_1 & \overline{a}_0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ \overline{a}_{n-1} & \overline{a}_{n-2} & \ldots & \overline{a}_0 \end{pmatrix},$$

$$M^{\zeta;f}_n := \begin{cases} \det\left(A^f_n + \tilde{A}^f_n\right), \ \textit{если}\, \zeta = \mathsf{c}\,, \\ \det\left(I_n - A^f_n \tilde{A}^f_n\right), \ \textit{если}\, \zeta = \mathsf{s}\,, \end{cases} \qquad n = 1, 2, \ldots,$$

где I_n – единичная $(n \times n)$–матрица. Тогда при $N = \infty, 0, 1, \ldots$ и $\zeta = \mathsf{c}, \mathsf{s}$ ряд $f(z)$ является рядом Тейлора функции класса $\mathcal{B}^{\zeta}_N$ тогда и только тогда, когда

$$M^{\zeta;f}_n > 0 \ \textit{при}\ n = 1, \ldots, N\,, \ \textit{и}\ M^{\zeta;f}_n = 0 \ \textit{при}\ n = N+1, N+2, \ldots$$

(если $N = 0$, то отсутствуют неравенства $M^{\zeta;f}_n > 0$, $n = 1, \ldots, N$, а если $N = \infty$, то отсутствуют равенства $M^{\zeta;f}_n = 0$, $n = N+1, N+2, \ldots$).

В 2020 году в [3] найдены необходимые и достаточные условия, при которых функция $F(z)$, заданная своими значениями с учетом кратностей в бесконечной последовательности точек $e_1, e_2, \ldots$ круга $\mathbb{D}$, допускает продолжение до функции

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Шура. Полученный в [3] критерий, а также найденный позже аналогичный критерий по отношению к функциям Каратеодори, будут сформулированы в докладе в терминах величин $M_{E_n}^{\mathtt{c};F}$ и $M_{E_n}^{\mathtt{s};F}$, где $E_n := \{e_1, \dots, e_n\}$, совпадающих соответственно с определителями Каратеодори $M_n^{\mathtt{c};F}$ и Шура $M_n^{\mathtt{s};F}$ в случае, когда $e_1 = e_2 = \dots = 0$ и $F(z) = \sum_{k=0}^{\infty} \frac{F^{(k)}(0)}{k!} z^k$.

Определение. *Пусть $F(z)$ – функция, определенная с учетом кратностей в бесконечной последовательности точек $e_1, e_2, \dots$ круга $\mathbb{D}$, т.е. если ν_n – кратность точки e_n в множестве $E_n := \{e_1, \dots, e_n\}$, то с учетом ранее определенных производных в точке e_n определена $(\nu_n - 1)$-я производная функции $F(z)$, которая обозначается через $F^{(\nu_n - 1)}(e_n)$.*

Функция $F(z)$ допускает продолжение до функции класса $\mathcal{B}_N^{\zeta}$ ($N = \infty, 0, 1, \dots$, $\zeta = \mathtt{c}, \mathtt{s}$), если и только если существует функция $\mathcal{F}_N^{\zeta}(z) \in \mathcal{B}_N^{\zeta}$ такая, что

$$\left(\mathcal{F}_N^{\zeta}\right)^{(\nu_n - 1)}(e_n) = F^{(\nu_n - 1)}(e_n) \text{ при всех } n = 1, 2, \dots ,$$

где ν_n – кратность точки e_n в множестве $E_n = \{e_1, \dots, e_n\}$.

Если $F(z)$ и $G(z)$ – две функции, определенные с учетом кратностей в точках множества E_n, то с учетом кратностей определены и функции $(F \pm G)(z)$, $(FG)(z)$, $(F/G)(z)$ (в последнем случае при условии $G(z) \neq 0$, $z \in E_n$).

Перестановкой элементов приведем множество $E_n = \{e_1, \dots, e_n\}$, $n = 1, 2, \dots$, к виду $E_n = \{e_1, \dots, e_1, \dots, e_k, \dots, e_k\}$, где $e_1, \dots, e_k$ попарно различны и имеют кратности $r_1, \dots, r_k$ соответственно, $r_1 + \dots + r_k = n$, и положим

$$A_{E_n}^F := \begin{pmatrix} \frac{(\varphi_0 F)(e_1)}{0!} & .. & \frac{(\varphi_0 F)^{(r_1-1)}(e_1)}{(r_1-1)!} & \dots & \frac{(\varphi_0 F)(e_k)}{0!} & .. & \frac{(\varphi_0 F)^{(r_k-1)}(e_k)}{(r_k-1)!} \\ .. & .. & .. & \dots & .. & .. & .. \\ \frac{(\varphi_{n-1} F)(e_1)}{0!} & .. & \frac{(\varphi_{n-1} F)^{(r_1-1)}(e_1)}{(r_1-1)!} & \dots & \frac{(\varphi_{n-1} F)(e_k)}{0!} & .. & \frac{(\varphi_{n-1} F)^{(r_k-1)}(e_k)}{(r_k-1)!} \end{pmatrix},$$

где $\varphi_p(z) := z^p$, $p = 0, 1, \dots$ ($z^0 := 1$, включая $z = 0$).

Введем в рассмотрение также и матрицу $\tilde{A}_{E_n}^F$, получаемую из матрицы $A_{E_n}^F$ комплексным сопряжением и записью строк и столбцов в обратном порядке, т.е. если $A_{E_n}^F = (a_{k,j})_{k,j=1,\dots,n}$, то $\tilde{A}_{E_n}^F = (\overline{a}_{n+1-k,n+1-j})_{k,j=1,\dots,n}$, и положим

$$M_{E_n}^{\mathtt{c};F} := \frac{\det \begin{pmatrix} A_{E_n}^{\varphi_0} & \tilde{A}_{E_n}^{\varphi_0} \\ -A_{E_n}^F & \tilde{A}_{E_n}^F \end{pmatrix}}{W_{E_n}} , \quad M_{E_n}^{\mathtt{s};F} := \frac{\det \begin{pmatrix} A_{E_n}^{\varphi_0} & \tilde{A}_{E_n}^F \\ A_{E_n}^F & \tilde{A}_{E_n}^{\varphi_0} \end{pmatrix}}{W_{E_n}} ,$$

где

$$W_{E_n} := \det \begin{pmatrix} A_{E_n}^{\varphi_0} & \tilde{A}_{E_n}^{\varphi_n} \\ A_{E_n}^{\varphi_n} & \tilde{A}_{E_n}^{\varphi_0} \end{pmatrix} , \; W_{E_n} \neq 0 \;, \text{ если } E_n \subset \mathbb{D} ,$$

Отметим, что каждая из величин $M_{E_n}^{\mathtt{c};F}$, $M_{E_n}^{\mathtt{s};F}$ вещественнозначна и инвариантна относительно перестановок точек множества E_n. В случае $e_1 = \dots = e_n = 0$ и $F(z) = \sum_{k=0}^{\infty} \frac{F^{(k)}(0)}{k!} z^k$ имеют место равенства $M_{E_n}^{\mathtt{c};F} = M_n^{\mathtt{c};F}$ и $M_{E_n}^{\mathtt{s};F} = M_n^{\mathtt{s};F}$.

При $N = \infty$ полученный многоточечный критерий по форме совпадает с классическим критерием Каратеодори–Шура, а именно, имеет место

Теорема 1. *Функция $F(z)$, заданная с учетом кратностей в точках $e_1, e_2, \dots$ круга $\mathbb{D}$, допускает продолжение до функции класса $\mathcal{B}_{\infty}^{\zeta}$, $\zeta = \mathtt{c}, \mathtt{s}$, тогда и только тогда, когда $M_{E_n}^{\zeta;F} > 0$, $n = 1, 2, \dots$.*

При $N \in \mathbb{Z}_+$ многоточечный критерий по форме немного отличается от классического и в двух крайних случаях ($e_1 = e_2 = \dots = e$ или $e_1, e_2, \dots$ попарно различны) имеет следующий вид.

Теорема 2. *Функция $F(z)$, заданная с учетом кратностей в точках $e_1, e_2, \dots$ круга $\mathbb{D}$, допускает продолжение до функции класса $\mathcal{B}_N^{\zeta}$, $N \in \mathbb{Z}_+$, $\zeta = \mathtt{c}, \mathtt{s}$, тогда и только тогда, когда $M_{E_n}^{\zeta;F} > 0$, $n = 1, \dots, N$, $M_{E_{N+1}}^{\zeta;F} = 0$ и*

$$\begin{cases} M_{E_{N+2p}}^{\zeta;F} = 0\,,\ p = 1, 2, \dots\,, \ \text{если } e_1 = e_2 = \dots = e\,, \\ M_{\{e_1,\dots,e_{N+1},e_{N+p}\}}^{\zeta;F^{1,\dots,N+1,N+p}} = 0\,,\ p = 2, 3, \dots\,, \ \text{если } e_1, e_2, \dots \ \text{попарно различны}\,, \end{cases}$$

где $F^{1,\dots,N+1,N+p}(z)$ – сужение функции $F(z)$ на множество $\{e_1, \dots, e_{N+1}, e_{N+p}\}$.

Обратим внимание на то, что достаточные условия, указанные в теореме 2 в случае $e_1 = e_2 = \dots = 0$ не содержат условий $M_{N+2p+1}^{\zeta;F} = 0$, $p = 1, 2, \dots$, присутствующих в достаточных условиях критерия Каратеодори–Шура (необходимость этих условий проверяется достаточно просто).

В общем случае при $N \in \mathbb{Z}_+$ формулировка многоточечного критерия сохраняет условия $M_{E_n}^{\zeta;F} > 0$, $n = 1, \dots, N$, $M_{E_{N+1}}^{\zeta;F} = 0$ и видоизменяет последующие условия (записываемые в виде равенств) при помощи введения дополнительных обозначений (подробнее см. [3]).

В завершение отметим, что имеет место следующая

Теорема 3. *Пусть $n \in \mathbb{N}$, $E_n = \{e_1, \dots, e_n\} \subset \mathbb{D}$, $F(z)$ – функция, заданная с учетом кратностей в точках множества E_n, такая, что $F(z) \neq -1$ при $z \in E_n$. Тогда для определенной с учетом кратностей в точках множества E_n функции $(T \circ F)(z) := \big(1 - F(z)\big)\big(1 + F(z)\big)$ имеют место равенства*

$$M_{E_n}^{\mathtt{C};T\circ F} = \frac{2^n}{\prod_{k=1}^n |1 + F(e_k)|^2} M_{E_n}^{\mathtt{S};F}\ ,\ M_{E_n}^{\mathtt{S};T\circ F} = \frac{2^n}{\prod_{k=1}^n |1 + F(e_k)|^2} M_{E_n}^{\mathtt{C};F}\,.$$

В частности, при $e_1 = e_2 = \dots = 0$ для ряда $F(z) = \sum_{k=0}^{\infty} a_k z^k$ имеем равенства

$$M_n^{\mathtt{C};T\circ F} = \frac{2^n}{|1 + a_0|^{2n}} M_n^{\mathtt{S};F}\ ,\ M_n^{\mathtt{S};T\circ F} = \frac{2^n}{|1 + a_0|^{2n}} M_n^{\mathtt{C};F}\,,$$

где $(T \circ F)(z)$ – ряд, получаемый формальным делением ряда $1 - F(z)$ на $1 + F(z)$.

В совокупности с утверждением (см. [2]) $f(z) \in \mathcal{B}_N^{\mathtt{S}} \iff (T \circ f)(z) \in \mathcal{B}_N^{\mathtt{C}}$ (за исключением функции $f(z) \equiv -1$, для которой $(T \circ f)(z) \equiv \infty$) теорема 3 означает, что критерии Каратеодори и Шура (и их многоточечные аналоги) эквивалентны друг другу в том смысле, что критерий Шура является следствием критерия Каратеодори и теоремы 3, и наоборот, критерий Каратеодори является следствием критерия Шура и теоремы 3.

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Harmonic foliations on a Riemann surface

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Гармонический слой $(\Omega, h):$ Ω – открытое множество на р.п., $h \in h(\Omega):$ $h:\Omega \to I := (a,b)$ *собственное* отображение, уровни $\Gamma_t : h = t$ компактные.

Обобщение теоремы Адамара о трёх кругах:

Предложение 1. $(\Omega,\, h)$ *– гармонический слой на р.п.* $\Rightarrow$ $\forall\ u \in sh(\Omega)$

$$M_u(t) := \int_{\Gamma_t} u\, d^c h \quad \text{и} \quad M_u^\infty(t) := \max_{\Gamma_t} u$$

– выпуклые функции на интервале I. А если $u \in h(\Omega)$, то M_u на I линейная.

- $h \in h(S)$, *отображение* $h : S \to [-\infty, a \le +\infty)$ *собственное,* $u \in sh(S) \Rightarrow$ *выпуклые функции* $M_u(t)$ *и* $M_u^\infty(t)$ *на* $(-\infty, a)$ *монотонно возрастают.*

* * * * *

$D \Subset S$ – область с регулярной границей на р.п. S.
$\circ \in D$, $G_D := G_D(\cdot, \circ)$ – функция Грина.

Линии Грина – ортогональные линиям уровня Γ_t интегральные кривые уравнения $d^c G_D = 0$ (определены вне крит.точек).
Σ – объединение всех крит. точек и линий Грина, которые начинаются в крит. точках.

$\int_{\Gamma_t} \partial G_D = \pi i \quad \Rightarrow \quad F(z) := \exp(-2 \int_\circ^z \partial G_D) \in \mathcal{O}(D \setminus \Sigma)$.

Предложение 2. F *– конформное отображение $D \setminus \Sigma$ на круг $\mathbb{D}$ с не более чем счётным локально конечным семейством радиальных разрезов.*

Следствие (Брело–Шоке). *Почти каждая регулярная линия Грина, имеют единственную предельную точку на ∂D. Почти все линии Грина имеют конечную длину относительно конформной метрики на S.*

MSC 30C15

Critical values of the finite Blaschke products[1]

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A finite Blaschke product of degree n is a rational function of the form

$$B(z) = e^{i\alpha} \prod_{k=1}^{n} \frac{z - z_k}{1 - \overline{z}_k z}, \qquad |z| < 1,$$

where α is a real number and $z_1, ..., z_n$ are complex numbers on the standard unit disk $|z| < 1$. For fixed $n \geq 2,\ B(0) = 0$ and $|B'(0)| = c,\ 0 < c < 1$, a sharp upper bound for the least critical values and a sharp lower bound for the greatest critical values of B are established. In proof of the upper bound, the structure of the Riemann surface of the function inverse to the function B, and the dissymmetrization of real functions [1] is essentially used. The exactness of this estimate is confirmed by the function

$$B_c(z) = z\frac{z^{n-1} - c}{1 - cz^{n-1}}.$$

Proof of the lower bound is based on the symmetrization approach [2], in which the result of symmetrization is located on the Riemann surface of the function inverse to a Chebyshev polynomial of the first kind of degree n. In this case, the extremal product is expressed through elliptic functions. The results obtained imply appropriate estimates for the critical values of complex polynomials of the form $P(z) = z^n + ... + cz$.

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Evolution families of holomorphic mappings with two fixed points

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In [1] Loewner obtained an equation that is satisfies by the evolution family of the semigroup of conformal mappings of the unit disk $\mathbb{D} = \{z \in \mathbb{C} \colon |z| < 1\}$ into itself leaving the origin fixed. In this paper, Loewner laid the fondations of the parametric method in geometric function theory and for the first time made progress in solving the coefficient problem stated as a conjecture by Bieberbach in 1916. For decades this problem determined the direction of development of the theory of univalent functions and was solved by de Branges in 1984 with the use of the Loewner parametric method.

Recently, models known as Stochastic Loewner Evolutions (or Schramm–Loewner–Evolutions) have become widespread. The remarkable discovery of Schramm [2] was that one can study Loewner equation with random driving functions. A large number of works have been devoted to the study of two types of stochastic Loewner evolutions (SLE), the chordal and the radial SLEs. The first of these models describes random curves that connect two boundary points of a simply connected planar domain, and the second describes random curves that extend from the boundary point towards a point located inside the domain. They correspond to two inequivalent normalizations of conformal mappings between simply connected domains in $\mathbb{C}$.

In this connection, the problem arises of studying evolution families of semigroups of holomorphic mappings of a simply connected domain into itself with fixed points. Let D be a proper simply connected domain in $\mathbb{C}$. Denote by $\mathfrak{P}(D)$ the set of all holomorphic mappings $f \colon D \to D$. Obviously, $\mathfrak{P}(D)$ forms a topological semigroup with respect to the operation of composition and the topology of locally uniform convergence in D. The role of the identity in this semigroup is played by the identity transformation $f(z) \equiv z$. Regarding $\mathbb{R}_+ = \{t \in \mathbb{R} \colon t \geq 0\}$ as an additive semigroup with the ordinary topology of the real numbers, we understand a one–parameter semigroup in $\mathfrak{P}(D)$ to be a continuous homomorphism $t \mapsto f^t$ acting from $\mathbb{R}_+$ to $\mathfrak{P}(D)$. As a rule, synthesis of algebraic and topological properties leads to fairly stringent constructions. This can be illustrated by the infinite differentiability with respect to t of the family of functions $f^t(z)$ of a one–parameter semigroup $t \mapsto f^t$ in $\mathfrak{P}(D)$. The function

$$v(z) = \frac{\partial}{\partial t} f^t(z)\Big|_{t=0} = \lim_{t \to 0} \frac{f^t(z) - z}{t}$$

is defined and holomorphic in D. We call this function the infinitesimal generator of the one–parameter semigroup $t \mapsto f^t$, or an infinitesimal transformation of the semigroup $\mathfrak{P}(D)$. The infinitesimal generator completely characterizes the one–parameter semigroup $t \mapsto f^t$ via the differential equation

$$\frac{\partial}{\partial t} f^t(z) = v(f^t(z))$$

and the initial condition $f^t(z)\big|_{t=0} = z$.

The extension of the construction of a one–parameter semigroup with the semigroup properties being preserved leads very naturally to the following concept.

Definition 1. *A two-parameter family $\{\varphi_{t,s} \colon 0 \leq s \leq t \leq T\}$ of the semigroup $\mathfrak{P}(D)$ will be called a backward evolution family in $\mathfrak{P}(D)$ on the interval $[0,T]$ if the following conditions hold:*

(*i*) $\varphi_{t,s}(z) = \varphi_{t,\tau} \circ \varphi_{\tau,s}(z)$ *for* $0 \leq s \leq \tau \leq t \leq T$;

(ii) $\varphi_{t,s}(z) \to z$ *locally uniformly in* D *as* $(t-s) \to 0$.

Definition 2. *A two-parameter family* $\{\psi_{s,t}\colon\ 0 \le s \le t \le T\}$ *of the semigroup* $\mathfrak{P}(D)$ *will be called a forward evolution family in* $\mathfrak{P}(D)$ *on the interval* $[0,T]$ *if the following conditions hold:*

(i) $\psi_{s,t}(z) = \psi_{s,\tau} \circ \psi_{\tau,t}(z)$ *for* $0 \le s \le \tau \le t \le T$*;*

(ii) $\psi_{s,t}(z) \to z$ *locally uniformly in* D *as* $(t-s) \to 0$.

In contrast to the case of a one-parameter semigroup, the differentiability of an evolution family in s and t does not follow directly from the definition. In the case of radial SLE, the unit disk $\mathbb{D}$ is chosen as the domain D, and mappings of evolution families leave the origin. In this case, the time parameter is chosen so that $\varphi'_{t,s}(0) = \psi'_{s,t}(0) = e^{s-t}$. This leads to the fact that evolution families are differentiable with respect to t and satisfy Loewner equations. In the case of chordal SLE, the upper half plane $\mathbb{H} = \{z\colon\ \mathsf{Im}z > 0\}$ is chosen as domain D, and the mappings satisfy hydrodynamic normalization. In this case, with an appropriate choice of time parameter the differentiability of evolution families is established [3].

Now, let f be a holomorphic mapping of the strip

$$\mathbb{S} = \{z \in \mathbb{C}\colon\ -\pi/2 < \mathsf{Im}z < \pi/2\}$$

into itself and $\mathsf{Re}f(x) \to \pm\infty$ as $x \to \pm\infty$. The existence of (finite or infinite) limits

$$d^{\pm}(f) = \lim_{x\to\pm\infty} \mathsf{Re}(x - f(x))$$

is established directly from the principle of the hyperbolic metric (see [4]). Moreover, $d^-(f) \le d^+(f)$ and for every $\alpha \in (0,1)$, in the strip $\mathbb{S}_\alpha = \{z \in \mathbb{C}\colon\ |\mathsf{Im}z| < \alpha\pi/2\}$, the following relations hold:

$$\lim_{\mathsf{Re}z\to\pm\infty} (z - f(z)) = d^{\pm}(f).$$

Denote by $\mathfrak{T}$ the semigroup of holomorphic mappings $f\colon\ \mathbb{S} \to \mathbb{S}$ such that $d^-(f) = 0$ and $d^+(f) < \infty$. The condition $z - f(z) \to 0$ as $\mathsf{Re}z \to -\infty$ is similar in form to hydrodynamic normalization adopted for conformal mappings of a half-plane. Note that any evolution family in $\mathfrak{T}$ can be transformed (by changing the time scale) into a normalized evolution family with $d^+(\varphi_{t,s}) = t-s$ and $d^+(\psi_{s,t}) = t-s$, $0 \le s \le t \le T$, (see [5]).

We let $\mathfrak{G}$ denote the class of functions h that holomorphic in $\mathbb{S}$ and admit the representation

$$h(z) = \lambda_1 \int\limits_{\mathbb{R}} \frac{d\mu_1(\theta)}{1 + ie^{\theta - z}} + \lambda_2 \int\limits_{\mathbb{R}} \frac{d\mu_2(\theta)}{1 - ie^{\theta - z}},$$

with some $\lambda_1, \lambda_2 \ge 0$, $\lambda_1 + \lambda_2 = 1$, and some probability measures μ_1, μ_2 on $\mathbb{R}$.

Theorem 1. *Let* $\{\varphi_{t,s}\colon\ 0 \le s \le t \le T\}$ *be a normalized backward evolution family in* $\mathfrak{T}$*. Then for any* $s \in [0,T)$ *and* $z \in \mathbb{S}$ *the function* $t \mapsto \varphi_{t,s}(z)$ *is absolutely continuous on* $[s,T]$ *and for almost all* t

$$\frac{\partial}{\partial t}\varphi_{t,s}(z) = -H(\varphi_{t,s}(z), t),$$

where $H(z,t)$ *is a function defined on* $\mathbb{S} \times [0,T]$ *holomorphic in* z*, measurable in* t*, and* $H(\cdot,t) \in \mathfrak{G}$ *for almost all* $t \in [0,T]$.

Theorem 2. *Let* $\{\psi_{s,t}\colon\ 0 \le s \le t \le T\}$ *be a normalized forward evolution family in* $\mathfrak{T}$*. Then for any* $s \in [0,T)$ *and* $z \in \mathbb{S}$ *the function* $t \mapsto \psi_{s,t}(z)$ *is absolutely* *continuous on $[s, T]$ and for almost all t*

$$\frac{\partial}{\partial t}\psi_{s,t}(z) = -\psi'_{s,t}(z)H(z,t),$$

where $H(z,t)$ is a function defined on $\mathbb{S} \times [0,T]$ holomorphic in z, measurable in t, and $H(\cdot,t) \in \mathfrak{G}$ for almost all $t \in [0,T]$.

Using these results, we can introduce a new type of SLE. Let $\gamma\colon [0,T] \to \overline{\mathbb{S}}$ be a continuous, simple curve. We will assume that γ satisfies the following conditions: $\mathsf{Im}\gamma(0) = -\pi/2$ and $\gamma(t) \in \mathbb{S}$ for every t. Let $S_t = \mathbb{S} \setminus \gamma[0,t]$ be the complement of the path up to time t. These assumptions ensure that S_t is a simply connected domain. Let $g_t\colon S_t \to \mathbb{S}$ be a conformal mapping. We can chose the time scale so that $f_t = g_t^{-1}$ belongs to $\mathfrak{T}$ and $d^+(f_t) = t$ for all t. Consider the forward evolution $\{\psi_{s,t}\colon 0 \le s \le t \le T\}$ in $\mathfrak{T}$ defined by

$$\psi_{s,t}(z) = g_s \circ f_t(z).$$

With this notations, $\psi_{0,t}(z) = f_t(z)$ and the function $g_t(\zeta)$ satisfies the equation

$$\frac{\partial}{\partial t}g_t(\zeta) = \frac{1}{1 + i\exp\{\theta(t) - g_t(\zeta)\}},$$

where $\theta(t)$ is a continuous function $\theta\colon [0,T] \to \mathbb{R}$.

Thus, we get an analogue of SLE in the strip $\mathbb{S}$:

$$\frac{\partial}{\partial t}g_t(\zeta) = \frac{1}{1 + i\exp\{U_t - g_t(\zeta)\}}$$

with $U_t = \sqrt{\kappa}B_t$ with B_t a standard Brownian motion and κ a real positive parameter.

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MSC 30F10

Ramified coverings of the Riemann sphere and uniformization[1]

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We study smooth one-parameter families of complex tori covering the Riemann sphere. The main problem is to describe such families in terms of projections of their branch-points. Earlier we investigated the problem for the case where, for every torus of the family, there is only one point lying over infinity [1], [2]. A similar problem for simply-connected compact coverings over the sphere was considered in [3]. Here we consider the general case.

Let $f_t(z) = f(z,t)$, $t \in [0,1]$, be a family of elliptic functions with periods $\omega_1 = 1$ and ω_2 describing a family of ramified coverings of the Riemann sphere by complex tori. We will name ω_2 the module of the corresponding torus $T = \mathbb{C}/\Omega$; here $\Omega = \{m + n\omega_2, m, n \in \mathbb{Z}\}$ is the lattice generated by ω_1 and ω_1. Every function of the family has the form

$$f_t(z) = c\int_{a_0}^{z} \frac{\prod_{j=0}^{N} \sigma^{m_j}(\xi - a_j)}{\prod_{l=0}^{P} \sigma^{n_l}(\xi - b_l)}\, d\xi + A_0.$$

Here we denote by σ the Weierstrass σ-function; a_j and b_l are critical points and poles of $f_t(z)$ with multiplicities $m_j + 1$ and $n_l - 1$; they, as long as the non-zero constant c, depend on t. Denote $A_j = f_t(a_j)$. If $A_j = A_j(t)$ are smooth functions with derivatives $\dot{A}_j$, then we show that the uniformizing functions satisfy the partial differential equation

$$\frac{\dot{f}(z,t)}{f'(z,t)} = \frac{1}{c}\sum_{k=1}^{N} F_k(z, a_1, \dots, a_N, b_1, \dots b_P)\dot{A}_j,$$

where F_k are definite meromorphic functions, and derive a system of ordinary differential equations for their critical points a_j, poles b_l, c, and module ω_2. Here $\dot{f}(z,t)$ and $f'(z,t)$ are partial derivative of $f(z,t)$ with respect to t and z.

Based on the system, we suggest an approximate method allowing to find an elliptic function uniformizing a given genus one ramified covering of the Riemann sphere. It consists in solving the Cauchy problem for a system of ODE describing the motion of critical points and poles. The initial data is taken from some torus with known uniformizing function $f(z,0)$. We give some examples illustrating the efficiency of the method.

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Further developments of the pluripotential theory

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1. Introduction. I remember, the classical potential theory is based on the class of subharmonic (sh) functions and on the Laplace operator Δ. The pluripotential theory, constructed in the 80-90s of the last century, is based on plurisubharmonic (psh) functions and on the Monge-Ampère operator

$$(dd^c u)^n = const\cdot \parallel \frac{\partial^2 u}{\partial z_j \partial \overline{z}_k} \parallel (dd^c|z|^2)^n.$$

Here as usual, $d = \partial + \overline{\partial},\ d^c = \frac{\partial - \overline{\partial}}{4i}$.

In the 1990s there were many attempts to develop and expand the pluripotential theory to broader classes of functions. One such class is the $m-$ subharmonic ($m-sh$) functions ($1 \le m \le n$). An upper semicontinuous in a domain $D \subset \mathbf{C}^n$ is said to be $m-$subharmonic in D, $u \in m-sh(D)$, if $dd^c u \wedge \beta^{m-1} \ge 0$, in the generalized sense, as current, i.e.

$$dd^c u \wedge \beta^{m-1}(\omega) = \int u\beta^{m-1} \wedge dd^c\omega \ge 0,\ \ \forall \omega \in F^{n-m,n-m},\ \omega \ge 0.$$

Here $\beta = dd^c|z|^2$ is the standard volume form of the space $\mathbf{C}^n$ and $F^{n-m,n-m}$ is the space of compactly supported in D, smooth differential forms, bi-degree $(n-m, n-m)$. Note that

$$psh(D) = 1 - sh(D) \subset m - sh(D) \subset n - sh(D) = sh(D).$$

Such functions have an excellent geometric characterizations.

Theorem 1. *(Z. Khusanov, B. Abdullaev, 1990, [10]). An upper semicontinuous function u, defined in a domain $D \subset \mathbf{C}^n$ is $m-sh$ if and only if for any complex plane $\Pi \subset \mathbf{C}^n$, $\dim \Pi = m$, the restriction $u|_\Pi \in sh\left(\Pi \cap D\right)$.*

$m - sh$ functions and functions closely related to them are considered and applied in various problems in the function theory (see [1],[2]). In a series of papers F.R.Harvey and H.B.Jr.Lawson (see, an example [8]-[9]) functions closely related to the $m - sh$ functions were applied in the problems of convex geometry, convex hull and minimal surfaces in calibrated geometry.

We note that contrary to the expectations, the operator $(dd^c u)^{n-m+1} \wedge \beta^{m-1}$ is not suitable for the construction of the potential theory in the class of $m - sh$ functions. In particular, the equation $(dd^c u)^{n-m+1} \wedge \beta^{m-1} = 0$ does not determine the maximal $m - sh$ functions. Moreover, for $1 < m < n$ the class $m - sh$ does not correspond to this operator, since the next necessary condition does not hold: $m - sh \not\subset \left\{(dd^c u)^{n-m+1} \wedge \beta^{m-1} \ge 0\right\}$. For example, $(dd^c u)^2 \wedge \beta = -\frac{\beta^3}{3} < 0$ for $u(z) = -|z_1|^2 + |z_2|^2 + |z_3|^2 \in 2 - sh\left(\mathbf{C}^3\right)$. So that the class $m - sh$ is not right for the operator $(dd^c u)^{n-m+1} \wedge \beta^{m-1}$. In this way B. Abdullaev suggested to use a subclass of the class $m - sh$:

$$(A)m - sh = \left\{u \in m - sh : (dd^c u)^{n-m+1} \wedge \beta^{m-1} \ge 0\right\} \subset m - sh$$

and Z.Błocki ([5], see also [6]) proposed using a class of functions

$$(B)m - sh =$$

$= \left\{u \in m - sh : (dd^c u) \wedge \beta^{n-1} \geq 0,\ (dd^c u)^2 \wedge \beta^{n-2} \geq 0, ..., (dd^c u)^{n-m+1} \wedge \beta^{m-1} \geq 0\right\}.$

Note that $(B)m - sh \subset (A)m - sh \subset m - sh$ and the class $(B)m - sh$ was found to be more flexible than $(A)m - sh$. In particular, in the class of $(B)m - sh$ functions was constructed the potential theory (see, Abdullaev-Sadullaev [3]), that means all the main potential properties of this class are proved.

Problem. Is $(A)m - sh = (B)m - sh$?

Theorem 2. *[4] If $u(z) \cap 2 - sh$ and $(dd^c u)^{n-1} \wedge \beta \geq 0$, then*

$$(dd^c u) \wedge \beta^{n-1} \geq 0,\ (dd^c u)^2 \wedge \beta^{n-2} \geq 0, ..., (dd^c u)^{n-1} \wedge \beta \geq 0,$$

i.e. $(A)m - sh = (B)m - sh$ for $m = 2$.

Theorem 3. *(S. Dinev [7]) $(A)m - sh = (B)m - sh$ for $n \leq 7$ and if $n \geq 8$, then $(A)m - sh \neq (B)m - sh$ even for $m = 3$.*

So, these two theorems fully explain the situation with the definitions of Abdullaev and Błocki.

2. Maximal $m - sh$ functions. One of the main objects of the class $m - sh$ functions are maximal functions, which are analogues of harmonic functions.

Definition 1. *A function $u(z) \in m - sh(D)$ is said to be maximal in a domain $D \subset \mathbf{C}^n$ if the maximum principle holds, that is, if $v \in m - sh(D) : \varliminf_{z \to \partial D} (u(z) - v(z)) \geq 0$, then $u(z) \geq v(z),\ \forall z \in D$.*

To find out the geometric nature of the maximal $m - sh$ functions, we first assume $u(z) \in C^2(D)$ and calculate $dd^c u \wedge \beta^{m-1}$ in terms of eigenvalues (at fixed point $z \in D$) of the complex Hessian $\left(\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}\right)$ of u, which is hermitian matrix. After a suitable unitary coordinate transformation, which does not change $\beta = dd^c|z|^2$, the operator $dd^c u$ can be written in the diagonal form:
$dd^c u = \frac{i}{2}[\lambda_1 dz_1 \wedge d\bar{z}_1 + ... + \lambda_n dz_n \wedge d\bar{z}_n]$, where the $\lambda_j = \lambda_j(z) \in \mathbf{R}^n$ are eigenvalues. We have

$$dd^c u \wedge \beta^{m-1} =$$

$$= \left(\frac{i}{2}\right)^m \sum_{1 \leq j_1 < ... < j_m \leq n} (\lambda_{j_1} + ... + \lambda_{j_m}) dz_{j_1} \wedge d\bar{z}_{j_1} \wedge ... \wedge dz_{j_m} \wedge d\bar{z}_{j_m}. \tag{1}$$

Positivity of the form $dd^c u \wedge \beta^{m-1}$, or in other words, $m-$ subharmonity of u means that all the coefficients (1) are positive,

$$\lambda_{j_1} + ... + \lambda_{j_m} \geq 0,\ 1 \leq j_1 < j_2 < ... < j_m \leq n.$$

We set

$$\mathcal{M}_u(z) = \left[\prod_{1 \leq j_1 < ... < j_m \leq n} (\lambda_{j_1}(z) + ... + \lambda_{j_m}(z))\right]^{\alpha_m}. \tag{2}$$

It is clear that $\mathcal{M}_u = \lambda_1 + ... + \lambda_n = \Delta u$ for $m = n$ and $\mathcal{M}_u = \lambda_1 ... \lambda_n \sim (dd^c u)^n$ for $m = 1$. The operator $\mathcal{M}_u$ is an operator in eigenvalues, symmetric, positive in the class $m - sh(D) \cap C^2(D)$.

Theorem 4. *(see [11]). The function $u(z) \in C^2 \cap m - sh(D)$ is maximal in the domain $D \subset \mathbf{C}^n$ if and only if $\mathcal{M}_u(z) \equiv 0$.*

3. The Dirichlet Problem. In this section, we will consider the Dirichlet problem for the equation

$$\mathcal{M}_u(z) = \psi(z),\ u(z) \in m - sh(D),\ u|_{\partial D} = \varphi(\xi),$$
$$\psi(z) \in C(\overline{D}),\ \varphi(\xi) \in C(\partial D),\ \psi(z) \geq 0 \tag{3}$$

We assume that the domain D is bounded, strictly $m-$ pseudoconvex, i.e., in a neighborhood of the closure $\overline{D}$ there exists a strictly $m - sh$ function $\rho(z)$ such that $d\rho|_{\partial D} \neq 0,\ D = \{\rho(z) < 0\}$.

Theorem 5. *(sf. [12]). If D is strictly $m-$ pseudoconvex and $\psi(z) \in C^\infty(\overline{D})$, $\psi(z) > 0$, $\varphi(\xi) \in C^\infty(\partial D)$, then (3) has a unique solution $u(z) \in m-sh(D) \cap C^\infty(\overline{D})$.*

Theorem 6. *(Maximum principle, sf. [12]). Let $u, v \in m - sh(D) \cap C^2(\overline{D})$: $\mathbf{M}_u(z) \leq \mathbf{M}_v(z)\ \forall z \in D$. Then $u|_{\partial D} \geq v|_{\partial D} \ \Rightarrow\ u|_D \geq v|_D$*

To construct a solution of the degenerate equation, case $\psi(z) \equiv 0$, we find the solutions

$$u_k \in C^\infty(\overline{D}),\ \mathcal{M}_{u_k}(z) = \frac{1}{k},\ u_k|_{\partial D} = \varphi.$$

Theorem 7. *The sequence $u_k(z)$ converges uniformly in $\overline{D}$. Its limit $u(z) = \lim\limits_{k\to\infty} u_k(z)$ is maximal function in D, $u(z) \in C(\overline{D})$, $u|_{\partial D} = \varphi$.*

This limit is naturally called the solution of the degenerated Dirichlet problem (3), for $\psi(z) \equiv 0$.

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MSC 47B10

Quantum differentials and function spaces[1]

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One of the goals of noncommutative geometry is the translation of basic notions of analysis into the language of Banach algebras. This translation is done using the quantization procedure which establishes a correspondence between function spaces and operator algebras in a Hilbert space H. The differential df of a function f (when it is correctly defined) corresponds under this procedure to the commutator of its operator image with some symmetry operator S which is a self-adjoint operator in H with square $S^2 = I$. The image of df under quantization is the quantum differential $d^q f$ of f which is correctly defined even for non-smooth functions f. The arising operator calculus is called the quantum calculus.

In our talk we shall give several assertions from this calculus concerning the interpretation of Schatten ideals of compact operators in a Hilbert space in terms of function spaces on the circle. The main attention is paid to the case of Hilbert–Schmidt operators. The role of the symmetry operator S is played in this case by the Hilbert transform. In the case of function spaces of several variables the symmetry operator may be defined in terms of Riesz operators and Dirac matrices.

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Geometric Transformations in Analysis: Dubinin's Road to Symmetry

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Since its invention by Jacob Steiner in 1838, symmetrization method had found numerous applications in mathematics and its applications. As well known, Steiner introduced his symmetrization when searching for a geometric tool to solve the classical isoperimetric problem on the maximal area for planar regions with fixed perimeter. After Steiner, many outstanding mathematicians, including George Pólya, Gabor Szegö, James Jenkins, Igor P. Mityuk, Al Baernstein II and others, introduced new symmetrizing procedures, each time targeting some unsolved extremal problem. Sometimes problems were challenging and resisted all known methods. If so, then some authors published their problems as open questions challenging mathematical community to find a way to solve them. Among these challenging problems were the following: G. Szegö problem on the covering of n symmetric rays, G. Pólya and G. Szegö problem on extremal properties of regular polygons, W. Hayman problem on covering of vertical intervals and two problems raised by A.A. Gonchar, one on the minimal capacity of condensers with plates of given length on the interval $[-1, 1]$ and the other problem on the maximal harmonic measure of n radial slits in the unit disk. It was clear to experts that solution of any one of these problems will require a new method.

It is amazing that all new methods needed to attack the above mentioned problems were suggested by just one outstanding mathematician of out time, by Professor Vladimir N. Dubinin. Among his methods are the following: the Separating transformation, Polarization and Dissymmetrization.

In the first part of this talk, I will introduce these transformations and discuss some their old applications to extremal problems in Complex Analysis and Potential Theory, which motivated the invention of these transformations. Then I will discuss my recent results on the the so-called "Sleeping armadillos problem", that is a problem on distribution of heat in systems of n balls, on the problem of distribution of heat on long pipes heated along some surface areas, and on the problem on the "squeezing function" that has application to spaces of analytic functions of one and several complex variables. My solution of each of these problems uses certain geometric transformation in Dubinin's style. Few remaining open and challenging extremal problems, which may require new symmetrization-type transformations will be also discussed.

MSC 2010: Primary 30C35; Secondary 30C55.

Internal geometry and boundary structure of a domain[1]

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Let Ω be a proper subdomain of the Euclidean space $\mathbb{R}^n$. Then the distance-ratio metric

$$j_\Omega(x,y) = \log\left(1 + \frac{|x-y|}{\min\{\operatorname{dist}(x,\partial\Omega), \operatorname{dist}(y,\partial\Omega)\}}\right)$$

plays an important role in Geometric Function Theory. For instance, the domain Ω is uniform if and only if the quasihyperbolic metric of Ω is bounded by a constant multiple of the distance-ratio metric of Ω. For disjoint non-empty compact sets E and F in the Möbius space $\overline{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\}$, the quantity

$$\delta(E,F) = \frac{\operatorname{dist}(E,F)}{\min\{\operatorname{diam} E, \operatorname{diam} F\}}$$

is convenient to estimate the conformal modulus of the family of curves joining E and F or the conformal capacity of the condenser $(E, \overline{\mathbb{R}}^n \setminus F)$. Being inspired by these quantities, we consider the quantity

$$J_\Omega(E) = \log\bigl(1 + \delta(E,F)\bigr), \quad E \subset \Omega, F = \overline{\mathbb{R}}^n \setminus \Omega.$$

One of our main results is the following [2].

Theorem 1. *Let $n = 2$ and Ω be a hyperbolic domain in the complex plane. Then, there is a constant $C > 0$ such that $J_\Omega(E) \le C h_\Omega(E)$ for arbitrary non-empty compact sets E in Ω, where $h_\Omega(E)$ denotes the hyperbolic diameter of E.*

We also present some results in [2] in the talk. This talk is based on joint work with Oona Rainio, Matti Vuorinen (Turku, Finland) and Tanran Zhang (Suzhou, China).

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MSC2010: 30C65 (26B35 31B15 46E35)

New results and current problems in quasiconformal analysis[1]

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A new concept [1] in the theory of homeomorphisms $\varphi : D \to D'$, $D, D' \subset \mathbb{R}^n$, of quasiconformal analysis is based on the fundamental connection between

1) a bounded composition operators $\varphi^* : L^1_p(D';\omega) \cap \mathrm{Lip}_l(D') \to L^1_q(D)$, $1 < q \leq p < \infty$, $n \geq 2$, from a weighted Sobolev space $L^1_p(D';\omega)$ to a weightlless one, where a weight $\omega : D' \to (0,\infty)$ belongs to $L_{1,\mathrm{loc}}(D')$, and $\varphi^*(f) = f \circ \varphi$ for $f \in \mathrm{Lip}_l(D')$;

2) estimates of the change in the capacity of the capacitors $\varphi^{-1}(E) = (\varphi^{-1}(F_0), \varphi^{-1}(F_1))$ in D through the weighted capacity of the capacitors $E = (F_0, F_1)$ in D':

$$\mathrm{cap}^{\frac{1}{q}}\big(\varphi^{-1}(E); L^1_q(D)\big) \leq \begin{cases} K_p \,\mathrm{cap}^{\frac{1}{p}}(E; L^1_p(D';\omega)), & \text{if } 1 < q = p < \infty, \\ \Psi(U \setminus F)^{\frac{1}{\sigma}} \,\mathrm{cap}^{\frac{1}{p}}\big(E; L^1_p(D';\omega)\big), & \text{if } 1 < q < p < \infty, \end{cases}$$

where K_p is a constant and Ψ is a quasiadditive set function;

3) homeomorphisms $\varphi : D \to D'$ of $W^1_{p,\mathrm{loc}}(D)$ and of finite distortion: $D\varphi(x) = 0$ a. e. on the set $Z = \{\det D\varphi(x) = 0\}$, with the operator distortion function

$$D \ni x \mapsto K^{1,\omega}_{q,p}(x,\varphi) = \begin{cases} \frac{|D\varphi(x)|}{|\det D\varphi(x)|^{\frac{1}{p}} \omega^{\frac{1}{p}}(\varphi(x))}, & \text{if } \det D\varphi(x) \neq 0, \\ 0, & \text{if } \det D\varphi(x) = 0, \end{cases}$$

belonging to $L_\sigma(D)$ where $\frac{1}{\sigma} = \frac{1}{q} - \frac{1}{p}$ if $1 \leq q < p < \infty$, and $\sigma = \infty$ if $q = p$.

Suffice it to say that this view of the problem contains practically all known approaches to different classes of mappings in quasiconformal analysis.

As an application we get some new assertions, for instance:

Theorem 1 [2]. *Let a homeomorphism $\varphi : D \to D'$ belong to Sobolev class $W^1_p(D)$ for $p > n-1$ at $n \geq 3$, or $p \geq 1$ at $n = 2$, and have the finite distortion. Then the inverse $\varphi^{-1} : D' \to D$ to such a homeomorphism belongs to the Sobolev class $W^1_{1,\mathrm{loc}}(D')$, has the finite distortion, and it is differentiable in D' a. e.*

The proof of Theorem 1 is new and it contains, as a special case, results by Hencl S. and Koskela P. in $\mathbb{R}^2$, and by Onninen J. in $\mathbb{R}^n$, $n \geq 3$. The method of proving does job also for obtaining similar result for mappings on Carnot groups.

Theorem 2. *If in above-mentioned line* 2) *we replace the capacity of capacitors by the modulus of families of curves we get the same class of mappings.*

Theorem 3. *All the smallest quantities* (*norms, constants and quasi-additive set functions*) *characterizing the behavior of the mappings in above-mentioned lines* 1)–3), *coincide.*

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MSC 30C85, 31A15, 65E10

Condenser capacity and domain functionals

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Conformal invariants related to natural geometric quantities such as extremal length of a curve family, hyperbolic geometry etc. play a role of "Rosetta's stone" in geometric function theory. One of these invariants is the conformal capacity $\mathrm{cap}(G;E)$ of a planar condenser $(G;E)$ where $G \subset \mathbb{R}^2$ is a domain and $E \subset G$ is compact [1]. Yet the value of the domain functional $\mathrm{cap}(G;E)$ is known only in a few special cases. Thus many researchers, motivated in part by the ideas of G. Pólya and G. Szegö [5], have studied simple domain functionals such as the length, area, perimeter of a set when looking for minorants or majorants for capacity. The progress of the computational methods has opened new avenues for this research. During the past two decades M. M.S. Nasser has developed numerical conformal mapping methods based on boundary integral equations and the fast multipole method. This talk is based on our very recent collaboration [2]-[4].

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Contributed Talks

MSC 30D15, 30E05, 42A38, 46F05

Properties of zero sets of functions invertible in the sense of Ehrenpreis in the Schwartz algebra [1]

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Let $\mathcal{E} = C^\infty(\mathbb{R})$ and $\mathcal{E}' = (C^\infty(\mathbb{R}))'$ be its strong dual space. The Fourier-Laplace transform $\mathcal{F}(S)$ of the distribution $S \in \mathcal{E}'$ is defined by $\mathcal{F}(S) = S(e^{-\mathrm{i}tz})$. The image $\mathcal{P} := \mathcal{F}(\mathcal{E}')$ is the linear space of all entire functions of exponential type having polynomial growth along the real axis. Being equipped with the topology induced from $\mathcal{E}'$, $\mathcal{P}$ becomes the topological algebra (*Schwartz algebra*).

The function $\varphi \in \mathcal{P}$ is *invertible in the sense of Ehrenpreis* if the primary ideal algebraically generated by φ in $\mathcal{P}$ is closed. It is also equivalent to the "division theorem": $\Phi \in \mathcal{P}$ *and* $\Phi/\varphi \in Hol(\mathbb{C})$ *imply* $\Phi/\varphi \in \mathcal{P}$.

In connection with the spectral synthesis problem for the differentiation operator D acting in $\mathcal{E}$ [1]–[3], it is interesting to study zero (sub)sets of invertible in the sense of Ehrenpreis functions in $\mathcal{P}$. We start with the following assertion due to L. Ehrenpreis [4, Proposition 6.1].

Proposition A. *Let* $\{(a_j; m_j)\}$ *denote the zero set of* $\psi \in \mathcal{P}$ *(here,* m_j *is the multiplicity of the zero* $a_j \in \mathbb{C}$*). If* ψ *is invertible in the sense of Ehrenpreis then* $\underline{\lim}_{j\to\infty} \frac{m_j}{|\mathrm{Im}\, a_j| + \ln|\mathrm{Re}\, a_j|} < \infty$.

Below, we present Theorem 1. It generalizes and refines the above assertion and also develops another result which we obtained in [5, Lemma 2].

Let $l : [0; +\infty) \to [1; +\infty)$ be non-decreasing function satisfying the conditions: $\ln t = O(l(t))$, $t \to \infty$, $\overline{\lim}_{t\to+\infty} \frac{\ln l(t)}{\ln t} < \frac{1}{2}$, $\overline{\lim}_{t\to+\infty} \frac{l(Kt)}{l(t)} < +\infty$ for some $K > 1$.

Theorem 1. *Let* $\psi \in \mathcal{P}$ *be invertible in the sense of Ehrenpreis with the zero set* $\mathcal{M} = \{\mu_k\}$*, and* $\mathcal{M}' \subset \mathcal{M}$ *be defined by* $\mu_k \in \mathcal{M}' \iff |\mathrm{Im}\, \mu_k| \le M_0 \cdot l(|\mathrm{Re}\, \mu_k|)$ *for a fixed* $M_0 > 0$.

Then,

$$\overline{\lim_{|x|\to\infty}} \frac{m_{\mathrm{Re}}(x,1)}{l(|x|)} < \infty,$$

where $m_{\mathrm{Re}}(x,1)$ *denotes the number of points of the sequence* $\mathrm{Re}\, \mathcal{M}' = \{\mathrm{Re}\, \mu_k : \mu_k \in \mathcal{M}'\}$ *contained in the segment* $[x-1; x+1]$.

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MSC 31B99

Extremal Decomposition Problems for p-Harmonic Robin Radii[1]

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The problems of extremal decomposition of plane domains have a rich history and are closely related to various problems in geometric function theory. In the works of Dubinin and his students, a series of results on the extremal decomposition for Robin radii was obtained. In particular, for disjoint plane subdomains D_1, D_2 of the unit disc U, points $a_1 \in D_1$, $a_2 \in D_2$, and closed sets $\Gamma_i \subset \partial D_i$ such that $(U \cap \partial D_i) \subset \Gamma_i,\ i = 1, 2$, the following inequality is true [1]

$$r(D_1, \Gamma_1, a_1) r(D_2, \Gamma_2, a_2) \leq \frac{|a_1 - a_2|^2 |1 - \overline{a_1} a_2|^2}{(1 - |a_1|^2)(1 - |a_2|^2)}.$$

Here $r(D, \Gamma, a)$ means the Robin radius of the domain D calculated at the point $a \in D$ relative to the set $\Gamma \subset \partial D$. A generalization of the Robin radius to the case of a spatial domain was given in the article [2], where the concept of the p-harmonic Robin radius $R_p(D, \Gamma, a)$ was introduced. In this talk, we discuss the results [3] on extremal decomposition dealing with the p-harmonic radii of domains in Euclidean space $\mathbb{R}^n$. As an example, we formulate the

Theorem. *Let the domain D be symmetric with respect to the hyperplane $\{x_1 = 0\}$, D_1 and D_2 are disjoint subdomains of D, points $\mathbf{a}_1 = (a_1, \ldots, a_n) \in D_1$, $\mathbf{a}_2 = (-a_1, \ldots, a_n) \in D_2$, are also symmetric with respect to $\{x_1 = 0\}$, $a_1 > 0$ for certainty. If the sets $\Gamma_i \subset \partial D_i$, $i = 1, 2$, satisfy the condition $(D \cap \partial D_i) \subset \Gamma_i$, then*

$$\mu_p(R_p(D_1, \Gamma_1, \mathbf{a}_1)) + \mu_p(R_p(D_2, \Gamma_2, \mathbf{a}_2)) \geq 2\mu_p(R_p(D^*, \Gamma^*, \mathbf{a}_1)),$$

where $D^* = D \cap \{x_1 > 0\}$, $\Gamma^* = \overline{D \cap \{x_1 = 0\}}$, and

$$\mu_p(t) = \begin{cases} -\log(t),\ p = n, \\ \frac{p-1}{n-p} t^{(p-n)/(p-1)}, \quad p \neq n. \end{cases}$$

In the case $\Gamma_i \subset (\partial D_i \cap \partial D)$, $i = 1, 2$, we have inequalities

$$\mu_p(R_p(D_1, \Gamma_1, \mathbf{a}_1)) + \mu_p(R_p(D_2, \Gamma_2, \mathbf{a}_2)) \leq 2\mu_p(R_p(D^*, \Gamma^{**}, \mathbf{a}_1)),\ p \leq n,$$

$$(-\mu_p(R_p(D_1, \Gamma_1, \mathbf{a}_1)))^{1-p} + (-\mu_p(R_p(D_2, \Gamma_2, \mathbf{a}_2)))^{1-p} \leq 2(-\mu_p(R_p(D^*, \Gamma^{**}, \mathbf{a}_1)))^{1-p},$$
$$p > n,$$

where $\Gamma^{**} = \partial D \cap \{x_1 \geq 0\}$.

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On the optimal shape of the gyrodine rotor

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A gyrodine or control moment gyroscope (CMG) is an orientation control device. It consists of a spinning rotor and one or more motorized gimbals that tilt the rotor's angular momentum. As the rotor tilts, the changing angular momentum causes a gyroscopic torque that reacts onto (rotates) the body to which the CMG is mounted.

The efficiency of the gyrodine is mainly determined by the rotor angular momentum $L = I\omega$ relative to the rotor axis of rotation.

Here I and ω are moment of inertia and speed of rotation of the gyrodine rotor relative to the axis of rotation. An increase in these parameters leads to an increase in internal stresses in the rotor material. Thus, the problem arises of choosing the rotation speed and shape of the gyrodine rotor in order to maximize the angular momentum L under the strength conditions, as well as restrictions on the mass m and the rotor dimensions (given by radius R). The strength condition is that the stresses at each point do not exceed the specified limit values.

The gyrodine rotor is a revolution body, the thickness of which depends only on the distance r to the axis of rotation, $r \in [0, R]$. The rotor surface is defined by rotating the curves $z = \pm z(r)$ around the axis.

In a spinning rotor, the strain field can be represented by the displacement function $u = u(r)$, which satisfies the stress-strain state equation

$$\frac{E}{1-\mu^2}\left(z\frac{du}{dr} + r\frac{dz}{dr}\frac{du}{dr} + rz\frac{d^2u}{dr^2} + \mu\frac{dz}{dr}u - z\frac{u}{r}\right) + \rho\omega^2 r^2 z = 0, \quad r \in (0; R),$$

equipped with boundary conditions $u(0) = 0$, $\frac{du}{dr}(R) + \mu\frac{u(R)}{R} = 0$. Here ρ — density, E — Young's modulus, μ — Poisson's ratio of rotor material. The fields of radial and tangential stresses are given by: $\sigma_r = \frac{E}{1-\mu^2}\left(\frac{du}{dr} + \mu\frac{u}{r}\right)$ and $\sigma_t = \frac{E}{1-\mu^2}\left(\mu\frac{du}{dr} + \frac{u}{r}\right)$.

The maximum mechanical stress corresponding to the stretching of elementary volumes in the body of a rotor with a given half-thickness $z(\cdot)$ and rotating with angular velocity ω is described by the functional:

$$\mathfrak{S}_{R,\rho,E,\mu}\big(\omega, z(\cdot)\big) = \sup_{x\in[0,R]} \max\big(\sigma_r(x), \sigma_t(x)\big)\,.$$

For a given material, mass m and radius R of the gyrodine rotor, it is required to choose its shape $z(r)$ in a given class of functions $\mathcal{F} \subset C^1[0, R]$ with an additional condition $4\pi\rho\int_0^R r\, z(r)\, dr = m$ and the angular velocity ω in order to maximize the angular momentum

$$L\big(\omega, z(\cdot)\big) = 4\pi\rho\omega\int_0^R r^3 z(r) dr,$$

taking into account the ultimate strength of the rotor material: $\mathfrak{S}_{R,\rho,E,\mu}\big(\omega, z(\cdot)\big) \leqslant [\sigma]$, where $[\sigma]$ denotes the maximum allowable stress of the rotor material.

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On the solvability of boundary problems for the inhomogeneous Schrodinger equation in the class of φ - equivalent functions on non-compact Riemannian manifolds[1]

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This paper investigates the solvability of boundary value problems for the inhomogeneous Schrödinger equation

$$Lu \equiv \Delta u - c(x)u = g(x) \qquad \text{on} \quad M, \tag{1}$$

where M is an arbitrary smooth connected noncompact Riemannian manifold without edge, the functions $c(x), g(x) \in C^{0,\alpha}(G)$, $c(x) \geq 0$, for any compact subset $G \subset M$, $0 < \alpha < 1$.

By a solution of equation (1) on a manifold M we mean the function $u \in C^2(G)$ satisfying this equation on any compact subset $G \subset M$. We will consider solutions of equation (1) on M with a given asymptotic behavior.

Let $\{B_k\}_{k=1}^{\infty}$ — an exhaustion of M, i.e. a sequence of precompact non-empty open subsets in M such that $\overline{B_k} \subset B_{k+1}$ and $M = \cup_{k=1}^{\infty} B_k$. Hereafter, we assume that the boundaries of ∂B_k are smooth submanifolds.

Definition 1. Let $\varphi \in C^0(M)$ and $\lim\limits_{k\to\infty} ||\varphi||_{C^0(M\setminus B_k)} = 0$. We will say that continuous functions f_1 and f_2 *φ-equivalent on M* if there exists a constant $C > 0$ such that

$$|f_1(x) - f_2(x)| \leq C \cdot \varphi(x),$$

for any $x \in M$.

The introduced relation is an equivalence relation, i.e. it has properties of reflexivity, symmetry and transitivity, and divides the set of all continuous functions on M into equivalence classes. Let us denote by $[f]_{\varphi}$ — the class of continuous functions which φ–equivalent on M to some continuous function f. The notion of φ-equivalence allows us not only to describe the asymptotic behavior of solutions of the boundary value problem for equation (1), but also to estimate the speed of its approach to boundary data at «infinity». It is more rigorous than the notion of equivalence functions introduced in [1] and [2].

Theorem. *Let M be an arbitrary smooth connected noncompact Riemannian manifold, $B \subset M$ an arbitrary connected compact subset and ∂B be a smooth subset.*

(i) Let u be a solution of equation (1) on M such that $u \in [f]_{\varphi}$. Then for every function $\Phi \in C^0(\partial B)$ there exists a solution u_0 of equation (1) on $M\setminus B$ such that $u_0 \in [f]_{\varphi}, u_0|_{\partial B} = \Phi$.

(ii) Let for every function $\Phi \in C^0(\partial B)$ there exist a solution u_0 of equation (1) on $M\setminus B$ such that $u_0 \in [f]_{\varphi}, u_0|_{\partial B} = \Phi$. Then on M there exists a solution u of equation (1) such that $u \in [f]_{\varphi}$.

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MSC 28A80

On fractal cubes with finite intersection property

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Let $n \geq 2$. A set $\mathcal{D} = \{\xi_1, \ldots, \xi_m\} \subset \{0, 1, \ldots, n-1\}^k$, $2 \leq \#D = m < n^k$, is called a *digit set*. The elements ξ_i of $\mathcal{D}$ define a system $\mathcal{S} = \{S_1, \ldots, S_m\}$ of similarities $S_i(x) = \dfrac{x + \xi_i}{n}$ in $\mathbb{R}^k$, mapping P^k to cubes with side $1/n$ which form a n^k partition of P^k. There is a unique non-empty compact set $F \subset P^k$ satisfying the equation

$$F = \bigcup_{i=1}^{m} S_i(F) = \frac{F + \mathcal{D}}{n},$$

which is called *a fractal k-cube* of the order n (see [2], [3])

A fractal 1-cube is called a fractal segment, a fractal 2-cube is a fractal square.

Intersections of a fractal k-cube F with *l-faces of the k-cube P^k* are called *l-faces* of F (for $0 \leq l < k$). Obviously, such l-faces are fractal l-cubes.

The partition cubes $P_i^k = S_i(P^k)$ of the k-cube P^k can intersect each other only by the images of respective pairs of opposite l-faces of P^k. Similarly, the copies $S_i(F) \subset P_i^k$ can intersect each other only by the images of respective pairs of opposite l-faces of F.

To verify that a fractal cube F has finite intersection property (see [1]), we need to study finite intersection conditions for pairs of fractal segments and fractal squares (the intersection of 0-faces is at most one-point).

Suppose $\mathcal{D}_1$ and $\mathcal{D}_2$ are the digit sets defining fractal segments K_1 and K_2 and suppose that $1 \in \mathcal{D}_1$ and $n-1 \in \mathcal{D}_2$. Then if $m \in \mathcal{D}_1 \cap (D_2 + 1)$, then the point $m/n \subset K_1 \cap K_2$. Such point is called a *transition point* for these fractal segments.

Theorem 1. *The fractal segments K_1 and K_2 of order n generated by digit sets $\mathcal{D}_1$ and $\mathcal{D}_2$ have a finite intersection in the following cases:*
1. $\#(\mathcal{D}_1 \cap \mathcal{D}_2) = 1$, and K_1 and K_2 have no transition points, then $\#(K_1 \cap K_2) = 1$;
2. $\mathcal{D}_1 \cap \mathcal{D}_2 = \varnothing$, and K_1 and K_2 have s transition points, then $\#(K_1 \cap K_2) = s$.

Theorem 2. *Two fractal squares K_1 and K_2 of order n, with digit sets $\mathcal{D}_1$ and $\mathcal{D}_2$ have a finite intersection in the following cases:*
1. $\#(\mathcal{D}_1 \cap \mathcal{D}_2) = 0$, there is at least one pair of vertex- or edge-adjacent copies $S_i(K_1) \subset K_1$ and $S_j(K_2) \subset K_2$ that have a non-empty finite intersection, and there is no such pair of edge-adjacent copies of K_1 and K_2 that intersect at an infinite number of points (see Theorem 1);
2. $\#(\mathcal{D}_1 \cap \mathcal{D}_2) = 1$, and any pair of vertex- or edge-adjacent copies of K_1 and K_2 has an empty intersection, then $\#(K_1 \cap K_2) = 1$.

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Ratios of the Gauss hypergeometric functions with parameters shifted by integers[1]

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Given real parameters a, b, c and integer shifts n_1, n_2, m, we consider the ratio

$$R(z) = \frac{{}_2F_1(a+n_1, b+n_2; c+m; z)}{{}_2F_1(a, b; c; z)}$$

of the Gauss hypergeometric functions. We find a formula for $\Im R(x \pm i0)$ with $x > 1$ in terms of some real hypergeometric polynomial P, beta density and the absolute value of ${}_2F_1(a, b; c; z)$. This allows us to construct explicit integral representations (related to the Stieltjes class $\mathcal{S}$) for the ratio R when the asymptotic behaviour at unity is mild and the denominator does not vanish.

Moreover, for arbitrary a, b, c and $\omega \leq 1$ the product $P(z+\omega)R(z+\omega)$ is shown to belong to the generalized Nevanlinna class $\mathcal{N}_\kappa^\lambda$ — one of natural generalisations of $\mathcal{S}$, see e.g. [3]. In the course of the proof we establish a few general facts relating generalized Nevanlinna classes to Jacobi and Stieltjes continued fractions, as well as to factorisation formulae for these classes.

Our findings are illustrated with several examples. In particular, we extend the results [1,2] by giving an in-depth analysis of the case $n_1 = 0$, $n_2 = m = 1$ known as the Gauss ratio.

This is a joint work with Dmitrii Karp.

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MSC 31C45, 46E35

Capacities of generalized condensers with A_1-Muckenhoupt weight

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The equality of capacity and modulus of condenser plays an important role in the geometric function theory. J. Hesse [1] proved this equality for condensers whose plates do not intersect with the boundary of domain. Further V.A. Shlyk [2] proved the equality p-capacity and p-modulus in general case. In [3] M. Ohtsuka established this result for capacities and moduli with A_p-Muckenhoupt weight, $p > 1$.

In 1999 H. Aikawa and M. Ohtsuka [4] introduced the concepts of capacities and moduli of vector measures and proved their equality. In particular, they proved the equality of two capacities of vector measures. This result implies the equality of capacity and modulus of condenser considered above.

We extend these results to the condensers in [5] for the case $p = 1$ and prove the equivalence of two capacities for Hesse condensers [1].

Theorem. For any condensers $(\{E_i\}, G, \{\delta_i\})$ as in [5],

$$C_{\mathcal{A},1}(\{E_i\}, G, \{\delta_i\}) = m_{\mathcal{A},1}(\alpha dH) = m_1(\alpha|\sqrt{\mathcal{B}}dH|).$$

Corollary. *If $w \in A_1$ then there exists a constant K such that for any Hesse condenser $(\{E_i\}, G, \{\delta_i\})$*

$$C_{\mathcal{A},1}(\{E_i\}, G, \{\delta_i\}) \le C^*_{\mathcal{A},1}(\{E_i\}, G, \{\delta_i\}) \le KC_{\mathcal{A},1}(\{E_i\}, G, \{\delta_i\}).$$

If the matrix $\mathcal{A}$ is continuous on G and $w \equiv 1$ then

$$C^*_{\mathcal{A},1}(\{E_i\}, G, \{\delta_i\}) = C_{\mathcal{A},1}(\{E_i\}, G, \{\delta_i\}).$$

These statements may be extend to the Carnot-Carathéodory spaces.

For more detailed explanation of notations and concepts see [4,5].

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Criteria for univalence and quasiconformal extension for harmonic mappings on planar domains

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This talk is about the extension of the theory of the Schwarzian derivative

$$Sf = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2$$

to planar harmonic mappings. We give criteria for univalence, homeomorphic and quasiconformal extension successively on quasidisks, finitely connected domains (all of whose boundary components are either points or quasicircles) and on uniform domains. These results are based on the definition of the Schwarzian derivative for harmonic mappings introduced by Hernández and Martín [*J. Geom. Anal.*; 2015].

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MSC 58J05

Massiveness of exterior of compact on non - compact Riemannian manifoldss [1]

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This work is dedicated to research of massive (D - massive) sets associated with stationary Shrodinger equation

$$\Delta u - q(x)u = 0 \tag{1}$$

on non - compact Riemannian manifolds. Here $q(x)$ is non negative smooth function on manifold.

Originally massive sets (D - massive) were introduced in [1] by A. Grigor'yan and via them criteria of existence of non - trivial harmonic functions (with finite energy integral) were obtained. Remarkable that massive sets gave opportunity to evaluate dimension of spaces of bounded harmonic functions (with finite energy integral) see [2].

Later A. Grigor'yan and A. Losev [3] obtained evaluation of dimension of spaces of solutions of stationary Shrodinger equation by generalizing of massive sets. Let $\Omega \subset M$ be open precompact set. We will call Ω q - massive if there is non - trivial subsolution of (1) on M, such as $u = 0$ on $M \setminus \Omega$ and $0 \leq u \leq 1$. If moreover

$$\int\limits_M |\nabla u|^2 + q(x)u^2 dx < \infty$$

then we will call Ω qD - massive.

Massive and q - massive sets have similar properties see [4]. For example, let $\Omega_1 \subset \Omega_2$. If Ω_1 is massive (q - massive) then Ω_1 is also massive (q - massive).

Let $F = M \setminus \Omega$ be compact. A. Grigor'yan [5] proved that Ω is massive if and only if Ω is D - massive. There is examples that such property of massive sets is not holds for q - massive sets. But it is possible to obtain following theorem.

Theorem 1. *If Ω is q - massive then there is nontrivial bounded solution of (1) on Ω with finite energy integral.*

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Application of semi-analytical hybrid method for the evaluation of the complex spectrum for waveguides with inhomogeneities [1]

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This paper presents a hybrid approach for modelling wave phenomena in multi-layered waveguides with internal delaminations (infinitesimally thin cuts) and multiple piezoelectric transducers and other surface-mounted obstacles. This problem cannot be solved using pure analytical methods due to employment of the governing equations for piezoelectric media and additional complex-shaped geometry of surface-mounted obstacles. Therefore, the hybrid approach employs the frequency domain spectral element method to discretize complex-shaped domains and the boundary integral equation method (BIEM) to simulate wave propagation in a multi-layered waveguide with a set of horizontal delaminations. The spectra of finite domains and waveguides with internal inhomogeneities have properties are investigated by the developed semi-analytical hybrid approach. In accordance with the boundary value problem, the components of the generalized state vector are assumed from the Sobolev space H^2 of square-integrable functions and their derivatives of orders $k < 2$. The variational formulation of the governing equations can be written using Lagrange interpolation polynomials at Gauss–Lobatto–Legendre points test functions, which belong to the space $W \in L^2$, where boundary conditions are automatically satisfied. The solution in a two-dimensional layered structure can be constructed using the BIEM as a superposition of the wave-fields $\boldsymbol{u}^{(m)}(\boldsymbol{x})$ scattered by inhomogeneities (delamination and surface-mounted obstacles) and induced by loads. The scattered and induced wave-fields have the following integral representations:

$$\boldsymbol{u}^{(m)}(\boldsymbol{x}) = \frac{1}{2\pi} \int\limits_{\Gamma} \mathbf{N}^{(m)}(\alpha, x_2) \boldsymbol{W}^{(m)}(\alpha) \mathrm{e}^{-\mathrm{i}\alpha x_1} \mathrm{d}\alpha. \tag{1}$$

Here $\boldsymbol{x} = \{x_1, x_2\}$, $\mathbf{N}^{(m)}(\alpha, x_2)$ is the Fourier transform of Green's matrix of m-th auxiliary boundary value problem and $\boldsymbol{W}^{(m)}(\alpha)$ is the Fourier transform of unknown crack opening displacement or surface load. The coupling of finite and unbounded domains is performed so that two solutions satisfy the boundary conditions in the contact areas, and the latter is reduced to a system of the linear algebraic equations with respect to the following vector of unknowns. Eigenfrequencies of the whole structure is then determined as zeros of the determinant of the left-hand side matrix using Muller's method. Some examples demonstrating changes in the spectrum are discussed.

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Lebesgue constants for Cantor sets [1]

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We evaluate the values of the Lebesgue constants in polynomial interpolation for three types of Cantor sets: uniformly perfect, with fast decrease of lengths of basic intervals, and for quadratic generalized Julia sets. In all cases, the sequences of Lebesgue constants are not bounded. This disproves the statement by Mergelyan (Theorem 6.2 in [1]).

The research is joint with Yaman Paksoy.

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MSC 30C85

Mappings with the property of bounded deformation of a harmonic measure

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In this report, we announce results concerning the plane mappings possessing the property of a bounded deformation of a harmonic measure.

Let D be a finitely connected plane domain and $\omega(z, E, D)$ denotes the harmonic measure of an arc $E \subset \partial D$ at the point $z \in D$. For more details and properties of harmonic measure see [1,2].

Let f be a sense-preserving diffeomorphism of the unit disk $\mathbb{D}$ onto domain $D \subset \mathbb{C}$, $D \neq \mathbb{C}$, that can be extended to a homeomorphism of the unit circle $\partial\mathbb{D}$ onto boundary curve ∂D in the sense of Caratheodory prime ends.

Definition. *We'll say that diffeomorphism f is a mapping with the property of bounded deformation of harmonic measure, if there exists some $C \geq 1$ such that for every subdomain $\Omega \subset \mathbb{D}$, for any arc $E \subset \partial\Omega$ and for any $z \in \Omega$*

$$\frac{1}{C}\omega(z, E, \Omega) \leq \omega(f(z), f(E), f(\Omega)) \leq C\omega(z, E, \Omega).$$

It is clear that any univalent conformal mapping f has this property with $C = 1$.

Theorem 1. *Let a mapping f has the property of bounded deformation of harmonic measure. Then f is C^4-quasiconformal in $\mathbb{D}$.*

Reminder that *pseudohyperbolic metric* in the unit disk $\mathbb{D}$ is defined by $\rho_{\mathbb{D}}(z_1, z_2) = \left|\frac{z_1 - z_2}{1 - \overline{z_1} z_2}\right|$. If D is an arbitrary simple connected domain in $\mathbb{C}$ and $D \neq \mathbb{C}$, the pseudo-hyperbolic distance is defined as $\rho_D(w_1, w_2) = \rho_{\mathbb{D}}(\Phi^{-1}(w_1), \Phi^{-1}(w_2))$, where Φ is an univalent conformal mapping of $\mathbb{D}$ onto D.

Theorem 2. *Let a mapping f has the property of bounded deformation of harmonic measure. Then f is C^2-quasi-isometry with respect to the pseudohyperbolic metrics $\rho_{\mathbb{D}}$ and ρ_D in the unit disk and domain D, i.e. for arbitrary $z_1, z_2 \in \mathbb{D}$*

$$\frac{1}{C^2}\rho_{\mathbb{D}}(z_1, z_2) \leq \rho_D(f(z_1), f(z_2)) \leq C^2\rho_{\mathbb{D}}(z_1, z_2).$$

The known Haiman-Wu theorem states [1] that for any univalent conformal mapping F of the disk $\mathbb{D}$ onto simple connected domain D a length $L(\gamma)$ of the preimage γ of intersection of domain D with an arbitrary straight line Γ is bounded by uniform constant:

$$L(F^{-1}(\Gamma \cap D)) \leq 4\pi.$$

Here we'll generalize the Haiman-Wu theorem.

Theorem 3. *Let a mapping f has the property of bounded deformation of harmonic measure and $D = f(\mathbb{D})$. Then for any straight line Γ*

$$L\left(f^{-1}(\Gamma \cap D)\right) \leq C^3 4\pi,$$

where L is an euclidean length of curve.

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MSC 26B10

Uniqueness theorem of reconstruction of pre image by its image under degenerate mapping[1]

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Let $f:\mathbb{R}^n\to\mathbb{R}^m$, $1<m<n$ be some differentiable mapping. We consider Lie group $\mathbb{G}$ of differentiable transformations of $\mathbb{R}^n$ and let $A_{\mathbb{G}}$ be its Lie algebra. It is well known that for each $X\in A_{\mathbb{G}}$ it corresponds vector field X^* in $\mathbb{R}^n$ such that $exp(tX)$ be local 1-parameter of local transformations of the vector field X^* (see, for example, Proposition 4.1 in [1]). We define the following set for each $X\in A_{\mathbb{G}}$

$$M_X=\{x\in\mathbb{R}^n : X^*(x)\in \mathrm{Ker}\ df\}.$$

Consider some finite set $P\subset\mathbb{R}^n$ such that $\forall X\in A_{\mathbb{G}}$ the set P is not subset of M_X.

Theorem 1. *There is some neighbourhood U of the identity $e\in\mathbb{G}$ such that if for some $g_1,g_2\in U\subset\mathbb{G}$ it holds*

$$f(g_1(P))=f(g_2(P))$$

then $g_1=g_2$.

For the proof of theorem 1 we use the following

Lemma 1. *Let $h(A,B)$ denotes Hausdorff distance between subsets $A,B\subset\mathbb{R}^m$. Consider curve $\gamma_t=exp(tX)$ for $X\in A_{\mathbb{G}}$. We put*

$$h(t)=h(f(P),f(\gamma_t(P))).$$

Then

$$\left|\frac{dh(t)}{dt}\Big|_{t=0}\right|=\max_{p\in P}\left|df(X^*(p))\right|.$$

In addition, we discuss the role of the obtained results in 3D reconstruction problems for objects on the single image. In particular, we note that in 2019 Baidu Robotics and Autonomous Driving Laboratory, together with Peking University, has set the task to solve problem determination of spatial position of vehicles on the road using data from a camera [2]. Also we note that some particular results of the considered problems have been obtained in [3],[4].

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MSC 31C45, 58E12

Variational Problems for Spatial Modules

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Let Σ be a family of $(n-1)$-dimensional surfaces separating the boundary components of the spherical ring $K = \{x : r < |x| < R\} \subset \mathbb{R}^n, n \geq 3$. Let Σ_Ω be a subfamily of Σ, consisting of surfaces enveloping an obstacle (continuum) $\Omega \subset K$ with given geometric parameters. The solution of the variational problem of finding a surface of revolution of the minimum area calculated in a special metric (see [1]) is used to obtain a nontrivial estimate for the p-module $M_p(\Sigma_\Omega)$ of the subfamilies Σ_Ω for $p \geq n-1$.

Consider an obstacle Ω, that lies in a layer bounded by concentric spheres of radii $r(1+l)$ and $r(1+L)$, $r \leq r(1+l) \leq r(1+L) \leq R$. Let $(n-1)$-dimensional measure of intersection of Ω with a sphere of radius τ is equal to $s_\Omega(\tau), r(1+l) \leq \tau \leq r(1+L)$. Let us denote by $\varphi_\Omega(\tau)$ the angle of the generator of the cone that cuts out on a sphere of radius τ a spherical cap with an area equal to $s_\Omega(\tau)$. We put $\varphi = \varphi(\Omega) = \min \varphi_\Omega(\tau), r(1+l) \leq \tau \leq r(1+L)$. Let $T(\theta) \subset K$ be a part of a cone with a generator angle equal to $\theta \geq \varphi(\Omega)$, lying in a layer between concentric spheres of radii $r(1+l)$ and $r(1+L)$. Let $\delta > 0$ and $T_1(\delta, \theta)$ be the part of $T(\theta)$ lying over the sphere of radius $r(1+l+\delta)$. If $l = 0$, we put $T_1(\delta, \theta) = \varnothing$. Accordingly, $T_2(\delta, \theta)$ is the part of $T(\theta)$ lying under a sphere of radius $r(1+L-\delta)$. If $r(1+L) = R$, we put $T_2(\delta, \theta) = \varnothing$. We set $T(\delta, \theta) = T_1(\delta, \theta) \cap T_2(\delta, \theta)$.

Theorem. *If* $\frac{1+L}{1+l} > e^{2h(\varphi)}$, *where* $h(\varphi) = \displaystyle\int_\varphi^{\frac{\pi}{2}} \frac{\sin^{n-2}\varphi}{\sqrt{\sin^{2(n-2)} t - \sin^{2(n-2)}\varphi}} dt$, *then*

$$M_p(\Sigma) - M_p(\Sigma_\Omega) \geq (\omega_{n-1})^{\frac{1}{1-n}} \int_{r(1+l)}^{r(1+L)} \frac{s_\Omega(\tau)}{\tau} d\tau + (\omega_{n-1})^{\frac{p}{1-n}} \int_{T(\delta,\theta)} |x|^{-p} dx,$$

for $\delta = (1+l)(e^{2h(\varphi)} - 1)$. *Here* ω_{n-1} *is the area of an* $(n-1)$*-dimensional sphere of unit radius. In particular,*

$$M_p(\Sigma) - M_p(\Sigma_\Omega) \geq \begin{cases} \frac{\varphi r^{n-p}}{\pi(\omega_{n-1})^{\frac{p-n+1}{n-1}}(n-p)} \left(I_n^p(0,\delta) + I_n^p(1,\delta)\right), & n \neq p; \\ \frac{\varphi}{\pi(\omega_{n-1})^{\frac{1}{n-1}}} \ln \frac{(1+L)(1+L-\delta)}{(1+l)(1+l+\delta)}, & n = p. \end{cases}$$

Here $I_n^p(u,v) = (1+L-uv)^{n-p} - (1+l+uv)^{n-p}$.

We note (see [1]) that

$$\sup_\varphi h(\varphi) = \frac{\pi}{2\sqrt{n-2}}.$$

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MSC 46E22, 30D10

Riesz bases of normalized reproducing kernels in radial Hilbert spaces of entire functions[1]

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The symbol stands H for a radial functional Hilbert space of entire functions stable with respect to dividing in the following sense:

1) all evaluation functionals $\delta_z : f \to f(z)$ are continuous;

2) if $F \in H$, $F(z_0) = 0$, then $F(z)(z - z_0)^{-1} \in H$.

The functional property of the space implies that it admits a reproducing kernel $k(\lambda, z)$: $f(z) = (f(\lambda), k(\lambda, z)), \quad \forall z \in \mathbb{C}, \quad \forall f \in H$.

A functional Hilbert space H is called radial if for some $F \in H$ and $\varphi \in \mathbb{R}$, the function $F(ze^{i\varphi})$ belongs to H, and $\|F(ze^{i\varphi})\| = \|F\|$.

Let $\kappa_z = k(\cdot, z)/\|k(\cdot, z)\|$ be the normalized reproducing kernel at z. We say that $\{\kappa_{z_i}\}_{i=1}^{\infty}$ a Riesz basis in H if it is complete and for some $C > 0$ and for each finite sequence $\{a_i\}_{i=1}^{n} \subset \mathbb{C}$, we have

$$\frac{1}{C}\sum_{i=1}^{n}|a_i|^2 \le \left\|\sum_{i=1}^{n} a_i\kappa_{z_i}\right\|^2 \le C\sum_{i=1}^{n}|a_i|^2.$$

Let the system of monomials $\{z_n\}_{n\in\mathbb{N}\cup\{0\}}$ is complete in H. And let $u(n) = \|z^n\|$, $n \in \mathbb{N} \cup \{0\}$, $u(t)$ be a piece-wise linear convex function with jumps at integer non-negative points. The Young conjugate function is $\widetilde{u}(x) = \sup_t(tx - u(t))$. Let

$$\rho_{\widetilde{u}}(x) = \sup\left\{t > 0 : \int\limits_{x-t}^{x+t} |\widetilde{u}'_+(\tau) - \widetilde{u}'_+(t)|d\tau \le 1\right\}.$$

The following theorems are proved.

Theorem 1. *The following conditions are equivalent:*

1) $\sup_p \inf_n(u'_+(n + p) - u'_+(n)) > 0$,

2) $\inf_{x\ge1} \rho_{\widetilde{u}}(x) > 0$.

And if these conditions hold true, then the space H possesses Riesz bases of normalized reproducing kernels.

Theorem 2. *If for some sequence $y_k \uparrow \infty$ the condition $\rho_{\widetilde{u}}(y_k) \to 0, \quad k \to \infty$, holds true, and there is a sequence $\gamma_k \uparrow \infty$, such that for some $a > 1$*

$$\frac{1}{a} \le \frac{\rho_{\widetilde{u}}(y_k)}{\rho_{\widetilde{u}}(x)} \le a,$$

for all $x \in (y_k - \gamma_k\rho_{\widetilde{u}}(y_k); y_k + \gamma_k\rho_{\widetilde{u}}(y_k))$, $k \in \mathbb{N}$, then there are no Riesz bases of normalized reproducing kernels in H.

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MSC 30E05, 14P10, 26C15

On boundary interpolation with finite Blaschke products[1]

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We denote the set of Blaschke products of degree m by

$$\mathbf{B}_m := \left\{ \gamma \prod_{j=1}^{m} \frac{z - a_j}{1 - \overline{a_j} z} \ : \ \gamma, a_1, \ldots, a_m \in \mathbb{C}, |\gamma| = 1, |a_1|, \ldots, |a_m| < 1 \right\}.$$

Here, $\mathbf{B}_0$ consists of constants (with modulus 1). We also write

$$\mathbf{B}_{\leq m} := \cup_{k=0}^{m} \mathbf{B}_k.$$

In this talk, we discuss an alternative proof of the Jones-Ruscheweyh theorem [1].

Theorem 1. *Let $0 \leq \varphi_1 < \varphi_2 < \ldots < \varphi_m < 2\pi$ and $\psi_1, \psi_2, \ldots, \psi_m \in [0, 2\pi)$. Then there exists a Blaschke product $B \in \mathbf{B}_{\leq m-1}$ such that $B(\exp i\varphi_j) = \exp i\psi_j$, $j = 1, 2, \ldots, m$.*

The approach is based on direct solution of the equations. To be more precise, the original equations are transformed into a system of polynomial equations with real coefficients. This leads to “geometric representation” of Blaschke products. Then, a Positivstellensatz by Prestel and Delzell [2] and a representation of positive polynomials in a special form due to Berr and Wörmann [3] together with a particular structure of the equations are used.

This is based on a joint work with Béla Nagy.

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MSC 58C35, 28C99

Coarea Formula for Functions on Two-Step Carnot Groups with Sub-Lorentzian Structure[1]

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We prove the coarea formula of the new type on sub-Lorentzian structures. These structures are sub-Riemannian generalization of well-known Minkowskii geometry. In its particular 4-dimensional case, the square of the vectors' length with the coordinates (x, y, z, t) equals

$$x^2 + y^2 + z^2 - t^2.$$

A two-step Carnot group is a connected simply connected stratified Lie group $\mathbb{G}$ whose Lie algebra V is graded, i.e., can be represented in the form $V = V_1 \oplus V_2$, where $[V_1, V_1] = V_2$ and $[V_1, V_2] = \{0\}$. Here we introduce the Minkowski-type structure in V_1, where the square of length along the vector field X_1 is negative.

Theorem 1. *Suppose that $\varphi : \Omega \to \mathbb{R}$ is a C^1-smooth function such that*

$$(X_1\varphi)^2 - \sum_{i=2}^{\dim V_1} (X_i\varphi)^2 \geq c > 0$$

everywhere; here $\Omega \subset \mathbb{G}$ is an open set. Then the coarea formula

$$\int\limits_{\Omega} \sqrt{(X_1\varphi)^2 - \sum_{i=2}^{\dim V_1} (X_i\varphi)^2}\, d\mathcal{H}^{\nu} = \int\limits_{\mathbb{R}} dz \int\limits_{\varphi^{-1}(z)} d\mathcal{H}^{\nu-1}_{\mathfrak{d}}$$

is valid, and the level sets are spacelike. Here the Hausdorff measure $\mathcal{H}^{\nu-1}_{\mathfrak{d}}$ is constructed on the level sets with respect to the sub-Lorentzian metric.

The result is new even for the classical Minkowski case $\mathbb{R}^{n+1}_1$.

Theorem 2. *On Minkowski geometry structures $\mathbb{R}^{n+1}_1$, the coarea formula is*

$$\int\limits_{\Omega} \sqrt{\Big(\frac{\partial\varphi}{\partial t}\Big)^2 - \sum_{i=1}^{n}\Big(\frac{\partial\varphi}{\partial x_i}\Big)^2}\, d\mathcal{H}^{n+1} = \int\limits_{\mathbb{R}} dz \int\limits_{\varphi^{-1}(z)} d\mathcal{H}^{n}_{\mathfrak{d}}.$$

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MSC 30C20

On the critical points of Mityuk's radius[1]

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We study the distribution and geometry of the critical points of the conformal radius

$$\Omega(w) = |f'(w)|/|\phi(w,w)| \tag{1}$$

where $F(w,\zeta) = (w-\zeta)\phi(w,\zeta)$ maps the domain D in the complex plane onto the canonical domain Δ; as a rule, Δ is the disc with some (empty or not) family of slits.

Over the simply connected regions the function (1) is the well-known inner mapping radius of Polya-Szegö. For multiply connected D's, the quantity (1) represents the Mityuk radius corresponding to the generalized reduced module, $M(w) = (2\pi)^{-1}\ln\Omega(w)$.

It is the critical points of the function $M(w)$ from the seminal paper [1] that determine the solutions of the exterior inverse boundary value problem with respect to the parameter s – the arc length of the unknown contour (see [2], [3]).

Choosing the parameters to which such a contour of the exterior problem is assigned – be it the above classical Gakhov's problem with s, the mixed problem with respect to (s,θ), etc. – we choose different types of canonical regions Δ, which leads to different types of Mityuk's radius $\Omega = \Omega_\Delta(w)$.

Let us denote by Λ_Δ the set of all critical points of the function Ω_Δ in D. We discuss, in particular, the following

Property. *Let D be $(n+1)$-ly domain, and let the function F maps D conformally onto the disc $\Delta = F(D)$ with n slits of the fixed type t. There exists a number n_t depending only on t and such that if $n \geq n_t$, then the set Λ_Δ is non-empty.*

We discuss also the analogue of this statement for the Szegö inner radius [4].

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MSC 35B50

Maximum principle for solutions of the modified Newtonian gravitational potential equation in an unbounded domain [1]

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The problem of dark (hidden) mass (DM) in astronomy and cosmology remains relevant since the 70s of the XX century, since a significant number of both observational and theoretical results require taking DM into account [1]. However, all efforts aimed at detecting DM particles have not been successful. An alternative approach can be the construction of a theory of modified Newtonian dynamics (MOND) [2], which has a number of problems, including mathematical ones. MOND field equations lead to a modified non-linear version Poisson's equations

$$\operatorname{div}\left(\frac{|\nabla u|\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = g(x), \tag{1}$$

where $g(x)$ is a continuous function defined in bfR^n and equal to zero outside some bounded domain D.

Let $u(x)$ be a bounded in $\mathbf{R}^n$ solution to equation (1). Let us introduce the notation $M(t) = \max_{|x|=t} u(x)$, $R_0 = \sup_{x\in D}|x|$. We have proved that for any $R_2 > R_1 > R_0$ there is a constant C for which the inequality

$$M(r) \le \frac{1}{\sqrt{2}}\int\limits_{R_1}^{r}\sqrt{\frac{C^2}{s^{2n-2}} + \frac{C}{s^{n-1}}\sqrt{\frac{C^2}{s^{2n-2}}+4}}\,ds + M(R_1)$$

for all $r \in [R_1, R_2]$. Wherein

$$C \le \left(\frac{M(R_2)-M(R_1)}{\ln\frac{R_2}{R_1}}\right)^2 \text{ for } n=3 \text{ and } C \le \left(\frac{M(R_2)-M(R_1)}{2(\sqrt{R_2}-\sqrt{R_1})}\right)^2 \text{ for } n=2.$$

In particular, for $x \in \mathbf{R}^n \setminus D$ and $n \le 3$, the inequality

$$u(x) \le \max_{\partial D} u$$

is true. It should be noted that for $n = 3$ this statement does not hold for the equation of the minimal surface.

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MSC 30A10

Differential inequalities for polynomials with a zero outside the unit disc

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There are a lot of books and articles devoted to polynomial inequalities (e.g., [1-3]). Denote $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. In 1930, S. N. Bernstein proved the following

Theorem A [1]. *Let f and F be polynomials such that* 1) $\deg f \le \deg F = n$, 2) $|f(z)| \le |F(z)|$ on $\partial\mathbb{D}$, *3) F has all its zeros in $\overline{\mathbb{D}}$.* Then for $z \in \mathbb{C} \setminus \mathbb{D}$

$$|f'(z)| \le |F'(z)|. \tag{1}$$

For a polynomial f of degree at most n and $\alpha \in \mathbb{C}$ denote

$$S_\alpha[f](z) = zf'(z) - n\alpha f(z).$$

For $R \ge 1$ let Ω_R denote the complement to the open disc with diameter $(\frac{R}{R+1}, \frac{R}{R-1})$.

In [3], V. I. Smirnov generalized Theorem A.

Theorem B [3, p. 356]. *Let $R \ge 1$. Let f and F be polynomials from Theorem A. Then for $|z| = R$ and $\alpha \in \Omega_R$*

$$|S_\alpha[f](z)| \le |S_\alpha[F](z)|. \tag{2}$$

Note that condition 2) in Theorems A and B can be replaced by the equivalent condition: $|f(z)| \le |F(z)|$ for $|z| \ge 1$. We do this in Theorem 1. Condition 3) from Theorem A on the zeros of the polynomial F became a key point in many further results on differential inequalities for polynomials. For results concerning the development of this subject see, for example, the historical survey in the introduction of [2]. We remove this requirement replacing it with a less strong one. It give us new differential inequalities improving Smirnov's and Bernstein's inequalities.

Theorem 1. *Let $R \ge 1$. Let f and F be polynomials such that 1) $\deg f \le$ $\le \deg F = n$, 2) $|f(z)| \le |F(z)|,\ z \in \mathbb{C} \setminus \mathbb{D}$, 3) z_0 is a unique zero of F lying in $\mathbb{C} \setminus \overline{\mathbb{D}}$, k is order of $z_0, 1 \le k \le n-1$. Then for $|z| = R$ inequality (2) takes place for $\alpha \in D_R$ (precise description of D_R will be given at the conference talk).*

Using Theorem 1, we also obtained an improvement of the Bernstein inequality for polynomials having a zero outside the unit disk.

Theorem 2. *Let f and F be polynomials from Theorem 1. Inequality (1) takes place for all $z : |z| = R \ge 1$ if R does not belong to the interval*

$$I = \left(\left(1 - \frac{k}{n}\right)|z_0| - \frac{k}{n}; \left(1 - \frac{k}{n}\right)|z_0| + \frac{k}{n}\right).$$

For every $R \in I$ there exist polynomials f and F, satisfying conditions of Theorem 1, such that (1) is not true on an arc of the circle $\{z \in \mathbb{C} : |z| = R\}$.

Theorem 2 allows to extend Theorem A and obtain the Bernstein inequality (1) not only in $\mathbb{C} \setminus \mathbb{D}$, but in a part of the unit disc.

Theorem 3. *Let f and F be polynomials such that*

1) $\deg f \le \deg F = n$,

2) all the zeros $z_1, ..., z_m$ of F belong to $\overline{\mathbb{D}}$, $|z_1| > |z_2| \geq ... \geq |z_m|$, z_1 is a zero of order $k \in \mathbb{N}$;

3) $|f(z)| \leq |F(z)|$ for $|z| \geq r$ and fixed $r \in [|z_2|, |z_1|)$.

Then (1) takes place for all $z : |z| \in [r, \infty) \setminus \Big(\left(1 - \frac{k}{n}\right)|z_1| - \frac{k}{n}r; \left(1 - \frac{k}{n}\right)|z_1| + \frac{k}{n}r\Big)$.

Using Theorem 1, we obtained sharp upper estimates for derivative of arbitrary polynomial in terms of the polynomial values. The estimates improve some known results, in particular, an estimate due to S.N. Bernstein. This result will be presented at the conference talk.

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On the uniqueness of the solution of the Beltrami equation with a closed level line.

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In [1] we established one result (Theorem 3, p. 108) on the admissible rate of convergence to zero of solutions of an equation of the form $\Delta u + c(x)u = 0$ at the ends of Riemannian manifolds with metrics of a special form.

We have established that in the two-dimensional case this result can be useful in solving problems of a slightly different type. In particular, we have established the following.

Let $D \subset \mathbb{C}$ be the domain in which the Beltrami equation is defined

$$w_{\overline{z}} = \mu(z) w_z, \tag{1}$$

where $\mu(z)$ is a measurable complex-valued function such that

$$\forall D' \Subset D \ \operatorname*{ess\,sup}_{D'} |\mu(z)| < 1.$$

A simple closed arc $\Gamma \subset D$ will be called a μ-circle if there exists an annular neighborhood $U_\Gamma \Subset D$ and a solution $\omega = \omega(z)$ homeomorphic in U_Γ equation (1) taking Γ to the circle $|\omega| = 1$. Let $\Sigma_t = \omega^{-1}(|\omega| = t)$.

Theorem 1. *Let two solutions $w_1(z)$, $w_2(z)$ of equation (1) be defined in U_Γ, and $w_1(z) = w_2(z)$ on Γ. Then if*

$$\lim_{t\to 1} \frac{1}{1-t} \int\limits_{\Sigma_t} |\operatorname{Re} w_2(z) - \operatorname{Re} w_1(z)||dz| = 0,$$

then $w_1(z)$ and $w_2(z)$ are related by a relation of the form $w_2(z) = w_1(z) + ia$, where $a \in \mathbb{R}$.

In this version of the uniqueness theorem, it is not required that the arc Γ be the boundary of some simply connected domain $D_\Gamma \Subset D$, and that the solutions $w_i(z)$ be homeomorphic.

Equation (1) is of interest for the theory of quasiconformal mappings and their geometric applications. In recent decades, many works have been devoted to them (see, for example, [2,3] and the bibliography there).

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MSC 30D10

A fundamental principle for unbounded domains[1]

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Let $\Lambda = \{\lambda_k, n_k\}_{k=1}^{\infty}$ be a sequence of different complex numbers λ_k and their multiplicities n_k. We assume that $|\lambda_k|$ does not decrease and $|\lambda_k| \to \infty$, $k \to \infty$. By $\Xi(\Lambda)$ we denote the set of limits of convergent sequences of the form $\{\overline{\lambda}_{k_j}/|\lambda_{k_j}|\}_{j=1}^{\infty}$ ($\overline{\lambda}$ is the complex conjugate of λ). The set $\Xi(\Lambda)$ is closed and it is contained in the unit circle $S(0,1)$. We put

$$\mathcal{E}(\Lambda) = \{z^n e^{\lambda_k z}\}_{k=1,n=0}^{\infty,n_k-1}.$$

Let $D \subset \mathbb{C}$ be a convex domain, and let

$$H_D(\varphi) = \sup_{z\in D} \operatorname{Re}(ze^{-i\varphi}), \quad \varphi \in [0, 2\pi]$$

be its support function. We set

$$J(D) = \{e^{i\varphi} \in S(0,1) : h(\varphi, D) = +\infty\}.$$

If D is bounded, then $J(D) = \emptyset$. In the case of an unbounded domain, the following situations are possible: 1) $J(D) = S(0,1)$, i.e. $D = \mathbb{C}$, 2) D is the half-plane $\{z \in \mathbb{C} : \operatorname{Re}(ze^{-i\varphi}) < a\}$ and $J(D) = S(0,1) \setminus \{e^{i\varphi}\}$, 3) D is a strip $\{z \in \mathbb{C} : b < \operatorname{Re}(ze^{-i\varphi}) < a\}$ and $J(D) = S(0,1) \setminus \{e^{i\varphi}, e^{i\varphi+\pi}\}$, 4) in all other cases, $J(D)$ is an arc of the unit circle which subtends an angle of at least π.

Let $H(D)$ be the space of analytic functions on the domain D with the topology of uniform convergence on compact sets $K \subset D$, and let $W \subset H(D)$ be a nontrivial closed subspace invariant with respect to the differentiation operator. Let $\Lambda = \{\lambda_k, n_k\}$ be the multiple spectrum of the differentiation operator on W. Then $\mathcal{E}(\Lambda)$ is the family of its eigenfunctions and generalized eigenfunctions on W. The main problem of the theory of invariant subspace is that of fundamental principle, i.e., representing any function $g \in W(\Lambda, D)$ as a series in the system $\mathcal{E}(\Lambda)$:

$$g(z) = \sum_{k,n=0}^{\infty,n_k-1} d_{k,n} z^n e^{\lambda_k z}. \tag{1}$$

The following theorem gives a criterion for an arbitrary function $g \in W(\Lambda, D)$ to be represented by a series of the form (1) which converges in the whole plane.

Theorem. *Given a sequence $\Lambda = \{\lambda_k, n_k\}$ and a convex domain D such that the system $\mathcal{E}(\Lambda)$ is not complete in $H(D)$, the following conditions are equivalent:*

1) $\Xi(\Lambda) \subseteq J(D)$ and $S_\Lambda > -\infty$;

2) each functions $g \in W(\Lambda, D)$ can be represented as a series (1) converging uniformly on the compact sets in the plane.

The characteristic S_Λ was introduced by A.S. Krivosheev (see [1])

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MSC 30C55

Domains of univalence of holomorphic self-maps of a disc with two fixed points[1]

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Let $\mathscr{B}$ be the set of holomorphic self-maps of the unit disc $\mathbb{D} = \{z \in \mathbb{C}\colon |z| < 1\}$. The problem of finding domains of univalence on subclasses of $\mathscr{B}$ is concidered. In [1], Landau indicated the sharp domain in which all functions in $\mathscr{B}$ with a given value of the derivative at the interior fixed point are univalent. In [2], Goryainov discovered the existence of domains of univalence for the class $\mathscr{B}$ of functions with two fixed points and conditions on the values of the angular derivatives at the boundary fixed points. This work is devoted to the development of these results.

Sharp domain of univalence on the class $\mathscr{B}$ of functions with interior and boundary fixed points and restriction on the value of the angular derivative at the boundary fixed point is found.

Theorem 1. *Let $\alpha \in (1, 4]$. If $f \in \mathscr{B}$ with $f(0) = 0$, $f(1) = 1$ and $f'(1) \leqslant \alpha$ (in the sense of the angular limit), then f is univalence in the domain*

$$\mathscr{D}(\alpha) = \left\{ z \in \mathbb{D}\colon \frac{\left|1 - 2z + |z|^2\right|}{1 - |z|^2} < \frac{1}{\sqrt{\alpha - 1}} \right\}.$$

For every domain $\mathscr{Y}$, $\mathscr{D}(\alpha) \subset \mathscr{Y} \subset \mathbb{D}$, $\mathscr{Y} \neq \mathscr{D}(\alpha)$, there exists a function $f \in \mathscr{B}$ with $f(0) = 0$, $f(1) = 1$ and $f'(1) \leqslant \alpha$ which is not univalent in the domain $\mathscr{Y}$.

Asymptotically sharp two-sided estimate of the domains of univalence on the class $\mathscr{B}$ of functions with two fixed diametrically opposite boundary points, an invariant diameter and restriction on the product of the angular derivatives at the boundary fixed points is obtained.

Theorem 2. *Let $\alpha \in (1, 9)$. If $f \in \mathscr{B}$ with $f((-1, 1)) = (-1, 1)$, $f(1) = 1$, $f(-1) = -1$ and $f'(1)f'(-1) \leqslant \alpha$ (in the sense of the angular limit), then f is univalence in the domain*

$$\underline{\mathscr{U}}(\alpha) = \left\{ z \in \mathbb{D}\colon \frac{|1 - z^2|}{1 - |z|^2} < \sqrt{1 + \left(\sqrt{k(\alpha) + 1} + \operatorname{sign}(25 - 9\alpha)\sqrt{k(\alpha)}\right)^2} \right\},$$

where

$$k(\alpha) = \frac{(9\alpha - 25)^2\,(183 - 37\sqrt{\alpha})}{128(\alpha - 1)(9 - \alpha)(85 - 12\sqrt{\alpha})}.$$

Theorem 3. *Fix $\alpha \in (1, 9)$. For each boundary point z_0 of the domain*

$$\overline{\mathscr{U}}(\alpha) = \left\{ z \in \mathbb{D}\colon \frac{|1 - z^2|}{1 - |z|^2} < \frac{2\sqrt[4]{\alpha}}{\sqrt{(3 + \sqrt{\alpha})(\sqrt{\alpha} - 1)}} \right\}$$

there exists a function $f \in \mathscr{B}$ with $f((-1, 1)) = (-1, 1)$, $f(1) = 1$, $f(-1) = -1$ and $f'(1)f'(-1) \leqslant \alpha$ (in the sense of the angular limit) whose derivative vanishes at z_0.

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MSC 30C70

The variational method in the theory of quasiconformal mappings

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Let $\mathbb{K}$ be the class K - quasiconformal mappings of f of the domain $D \in \overline{\mathbb{C}}$ onto itself, normalized by the condition $f(\infty) = \infty$. In the classical work of Schiefer and Schober ([1]), a variational method was developed for constructing fundamental solutions for uniformly elliptic systems of equations, in which the formulated problem is reduced to a variational one on the class of K - quasiconformal mappings.

We consider a generalization of the classical result of Schiefer and Schober to the case of degenerate elliptic systems. Let $\mathbb{K}_\infty$ be the class of quasiconformal mappings that degenerate at the endpoint and onto ∞. Regarding the characteristics of such mappings, it is assumed that for them the integral of the following form $\int\limits_{r_0}^{r} \frac{1-q(\rho)}{1+q(\rho)} \frac{1}{\rho}\, d\rho$ is divergent at $r \to 0$. It is established that the class of such mappings is not empty. In addition, an integral equation is indicated, the solution of which determines the characteristics with the indicated properties.

On the class $\mathbb{K}_\infty$ we consider the functional

$$\Phi(z_0, z_0, f) = \lim_{r \to 0} \sqrt{\frac{A(z_0, z_0, f)}{\pi}} \left(exp\{ \int\limits_{r_0}^{r} \frac{1-q(\rho)}{1+q(\rho)} \frac{1}{\rho}\, d\rho \} \right)^{-1}.$$

The main result of our research is the following theorem.

Theorem 1. *Let f - extremal function for the variational problem of finding $\max\limits_{\mathbb{K}_\infty} \Phi(z_0, z_0, f)$, moreover, the functional $\Phi(z_0, z_0, f)$ has an essentially nonzero Gateaux derivative $L(\Psi(f(z_0, f))$. Then f is a solution of the following degenerate elliptic system:*

$$f_{\overline{z}} = k(z) \frac{|L(\Psi(f(z_0, f))|}{L(\Psi(f(z_0), f))} = \overline{f(z)} a.e.$$

where $k(z) = \frac{-1 + \int\limits_{r_0}^{r} \frac{1-q(\rho)}{1+q(\rho)} \frac{1}{\rho}\, d\rho}{1 + \int\limits_{r_0}^{r} \frac{1-q(\rho)}{1+q(\rho)} \frac{1}{\rho}\, d\rho}$.

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Some aspects of potential theory for elliptic equations on Riemannian manifolds

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The paper is devoted to the development of capacitary techniques in the study of harmonic functions. The deep connection between classical problems of the theory of functions of a complex variable, the theory of partial differential equations, the geometry of Riemannian manifolds, and the so-called geometrical analysis was pointed out in many researches in the last decades.

The origins of the topic are linked to classification theory of Riemannian surfaces. Distinctive property of two dimension surfaces of parabolic (hyperbolic) type is fulfillment (unfulfillment) theorem of Liouville, which claims that any positive superharmonic function on this surface is constant. This property is main in order to define parabolic and hyperbolic type manifolds with dimension higher than two. It is said that non-compact Riemannian manifold M has parabolic type if on M any positive superharmonic function is identically constant. Otherwise, it is said that M has hyperbolic type.

There are series of works which dedicated to learning properties of parabolic type manifolds. We should notice that the most effective technique in this direction is capacitive methods. A.A. Grigor'yan (1985) proved that complete manifold M has parabolic type if and only if capacity of any compact on M is equal to zero.

Developing this approach A.A. Grigor'yan (1990) obtained an estimate of the dimension of the space of bounded harmonic functions. Namely, it was proved that the dimension of the space of bounded harmonic functions on M is no less than $m \geq 2$ if and only if there are m pairwise disjoint massive subsets in M.

The following notion of *massiveness* play an important role in the sequel [1].

Let $\Omega \subset M$ be an open set. We say that a function $v \geq 0$ is an admissible subharmonic function for Ω if it is a bounded subharmonic function on M such that $v = 0$ in $M \setminus \Omega$ and $\sup_\Omega v > 0$. An open set Ω is called *massive* if there is at least one admissible subharmonic function.

The *subharmonic potential* b_Ω of an open set Ω is the supremum of all admissible subharmonic functions v for Ω such that $v \leq 1$.

Choose an exhaustion $\{B_k\}$ of M with smooth boundaries, and solve the following Dirichlet problem

$$\begin{cases} \Delta h_k(x) = 0, x \in B_k \\ h_k|_{\partial B_k} = v_\Omega|_{\partial B_k} \end{cases}.$$

The function

$$h_\Omega = \lim_{k\to\infty} h_k$$

is called the harmonic potential of the Ω. Let $HB(M)$ be a space of bounded harmonic function defined on M and $HG(M)$ be a linear hull of the set of harmonic potentials on M. The main result of the paper is the following theorem.

Theorem. *Space $HG(M)$ is dense in $HB(M)$.*

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MSC 33C45

Classical multiple orthogonal polynomials of a discrete variable

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Let $\mu_1, \dots, \mu_r$ be the positive Borel measures on $\mathbb{R}$ with finite moments. A polynomial P_n is called a *multiple orthogonal polynomial* if $P_n \not\equiv 0$, $\deg P_n \le rn$ and the following orthogonality relations are satisfied

$$\int P_n(x) x^k d\mu_j(x) = 0, \quad k = 0, \dots, n-1, \quad j = 1, \dots, r. \tag{1}$$

Such polynomials have a wide range of applications from number theory to random matrices.

In the case $r = 1$ we obtain the usual sequence of orthogonal polynomials. Classical orthogonal polynomials (Hermite, Laguerre, and Jacobi) can be introduced by *the Pearson equation* for the weight of orthogonality $\rho(x) := d\mu(x)/dx$:

$$\frac{d}{dx}\big(\sigma(x)\rho(x)\big) = \tau(x)\rho(x) \tag{2}$$

with polynomials σ, τ of degrees $\deg \sigma \le 2$ and $\deg \tau = 1$.

Orthogonal polynomials of a discrete variable correspond to the orthogonality measure $\mu(x) = \sum_{x \in S} \rho(x)\delta(x)$ for a discrete set S. Classical discrete orthogonal polynomials (Charlier, Kravchuk, Meixner and Hahn) can be introduced by the *difference* Pearson equation for $\rho(x)$.

There are two ways to determine classical multiple orthogonal polynomials in the continuous case. The first one [1] is when the supports $\operatorname{supp} \mu_j$ are coincide intervals, and the weights ρ_j satisfy the Pearson equation (2) with distinct τ_j. In this case, the restrictions for the degrees of σ and τ_j remain the same: $\deg \sigma \le 2$ and $\deg \tau_j = 1$.

There is also another way [2], which we consider in the particular case $r = 2$. Here, the weight functions ρ_1, ρ_2 are proportional to restrictions of a certain analytic function ρ to disjoint intervals, that is: $\rho_j := c_j \rho|_{\operatorname{supp} \mu_j}$ for a constant c_j. The function ρ is additionally assumed to satisfy the Pearson equation (2) with the polynomials σ and τ of degrees $\deg \sigma \le 3$ and $\tau = 2$.

Until recently, only the first way of constructing [3] multiple orthogonal polynomials of a discrete variable was applied. We succeeded in constructing classical discrete multiple orthogonal polynomials in the second way and to give their classification. The key idea is to choose integer lattices with shifts as discrete supports of orthogonality measures μ_j. We have shown that the resulting polynomials are given by Rodrigues difference formulas and satisfy high-order differential and recurrent equations.

This is a joint work with A. Dyachenko (University of Konstanz, Germany), arXiv:1908.11467.

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MSC 30D10

Functions of infinite order in upper half-plane[1]

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In the work [1] were considered the entire functions which zeros lie on the finite system of rays. In particular, it was proved that if f is the entire function of infinite order with positive zeros then its lower order also equals infinity. We prove a similar statement for analytic functions on the half-plane. The analytic function f in $\mathbb{C}_+ = \{z : \Im z > 0\}$ is called *proper analytic* if $\limsup_{z\to t} \log|f(z)| \le 0$ for all real numbers $t \in \mathbb{R}$. Proper analytic functions $f(z)$ have the following properties:

a) $\log|f(z)|$ has non-tangential limits $\log|f(t)|$ almost everywhere on the real axis and $\log|f(t)| \in L^1_{loc}(-\infty, \infty)$;

b) there exists a measure of variable sign ν on the real axis such that

$$\lim_{y\to+0} \int_a^b \log|f(t+iy)|\,dt = \nu([a,b]) - \frac{1}{2}\nu(\{a\}) - \frac{1}{2}\nu(\{b\})\,.$$

The measure ν is called *the boundary measure* of f;

c) $d\nu(t) = \log|f(t)|\,dt + d\sigma(t)$, where σ is a singular measure with respect to Lebesgue measure.

For a proper analytic function f, we define *the full measure* λ as

$$\lambda(K) = 2\pi \int_{\mathbb{C}_+\cap K} \Im\zeta\, d\mu(\zeta) - \nu(K)\,,$$

where μ is the Riesz measure of $\log|f(z)|$. The full measure of the proper analytic function is a positive measure, which explains the term "proper analytic function".

The main result is the following theorem.

Theorem 1. *If f is the proper analytic function in half-plane $\mathbb{C}_+$ of infinite order with the zeros on the finite system of rays*

$$\mathbb{L}_k = \left\{ z : \arg z = e^{i\theta_k},\ \theta_k = \frac{\pi p_k}{q_k} \right\};$$

$k \in \overline{1, N_0}$; $p_k, q_k, N_0 \in \mathbb{N}$; $p_k < q_k$; *then its lower order equals infinity.*

To prove this theorem, we used the Fourier series method from [2].

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MSC 31A15, 30E10

Equilibrium approximation of analytic functions

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1. Let be given K – a bounded continuum with a connected complement. Denote $\mathcal{P}_n(K)$ the set of complex products

$$\pi_k(z) := \Pi_{j=1}^k (z - z_j)^{a_j},\ a_j > 0, j = 1, 2, ..., k,\ a_1 + ... + a_k = 1,$$

admitting non-algebraic singularities with at most n zeros on K.

We introduce the uniform norm $\|\pi_n(z)\|_K := \max_{z\in K} |\pi_n(z)|$ on the class $\mathcal{P}_n(K)$ and consider the problem of finding in $\mathcal{P}_n(K)$ the element least deviating from zero by K; the problem is similar to the Chebyshev problem, with the difference that the solution is sought in a "wider" class.

Problem $\mathbb{E}_n(K)$. *We need to find*

$$\varrho_n(K) = \inf \{\|\sigma\|_K : \sigma \in \mathcal{P}_n(K)\},$$

and the functions $\pi_n \in \mathcal{P}_n(K)$, for which the infimum is attained.

2. Let the function

$$f(z) = z + f_0 + \frac{f_1}{z} + \frac{f_2}{z^2} + ...$$

be set by a power series in the infinite and unlimited continues along any path lying in $\mathbb{C}\backslash\Gamma$ such that every point of Γ is a singular point of f, including can be a branch point. The set of such functions is denoted by $\mathcal{A}(\mathbb{C}\backslash\Gamma)$. Let $f \in \mathcal{A}(\mathbb{C}\backslash\Gamma)$, denote $C_\beta(f) = \{z \in \mathbb{C} :\ |f(z)| = \beta\}$ and β^* the lower bound of such β that f is holomorphic and single-valued in $\operatorname{ext} C_\beta(f)$. Then, for $\beta \geq \beta^*$, the function f conformally and univalently maps the area $\operatorname{ext} C_\beta(f)$ to the exterior of the circle of the radius β centered at the origin.

Consider the class of compacts defined by the function f:

$$K_\beta(f) = \overline{\operatorname{int} C_\beta(f)},$$

where the dash at the top means the closure of the area, and $\operatorname{int} C_\beta(f)$ is the area bounded by the curve $C_\beta(f)$. Then it is clear that if $\beta > \operatorname{cap}(\Gamma)$, then the compact $K_\beta(f)$ is not empty, because it contains Γ, and for $\beta > \beta^*$, the compact $K_\beta(f)$ is simply connected and has an analytic boundary. We will say that the compact Γ satisfies the *minimality* condition for f if there is no compact Γ' and no analytic function f' such that $\operatorname{cap}(\Gamma') < \operatorname{cap}(\Gamma)$ and $f = f'$ in $\mathbb{C}\backslash\Gamma$.

Theorem 1. *If $f \in \mathcal{A}(\mathbb{C}\backslash\Gamma)$, Γ is a compact that does not split the plane and satisfies the minimality condition for f, $K_\beta(f)$ is a compact for some $\beta = cap(K_\beta(f))$, $\beta > cap(\Gamma)$ and π_n – a sequence of solutions to the problem $\mathbb{E}_n(K_\beta(f))$, for $n = 1, 2, ...$, then π_n uniformly inside $\mathbb{C}\backslash\Gamma$ converges to the function f:*

$$\pi_n(z) \rightrightarrows f(z),\ z \in \mathbb{C}\backslash\Gamma.$$

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UDC 517.95

On the solvability of boundary value problems for an inhomogeneous Schrodinger equation under variations of its coefficients on noncompact Riemannian manifolds[1]

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In this paper we study the problem of preserving the solvability of some boundary value problems for the inhomogeneous Schrodinger equation:

$$Lu \equiv \Delta u - c(x)u = g(x) \tag{1}$$

under changing its coefficients, on an arbitrary smooth connected non-compact Riemannian manifold M. We will assume that $0 \le c_1(x) \le c(x)$, $g_1(x) \le g(x) \le g_2(x)$, where $c(x),\ c_1(x), g(x),\ g_i(x) \in C^{0,\alpha}(G)$ for anyone $G \subset\subset M$, $0 < \alpha < 1$, $i = 1, 2$, $c(x) \not\equiv 0,\ c_1(x) \not\equiv 0$. By the solution of equation (1), we mean the function $u \in C^2(G)$ that satisfies this equation on each compact subset $G \subset\subset M$.

Together with equation (1) consider the equations

$$L_1 u \equiv \Delta u - c_1(x)u = g(x), \qquad Lu = g_1(x) \qquad \text{and} \qquad Lu = g_2(x).$$

Let $f_1(x)$ and $f_2(x)$ be arbitrary continuous functions on M, and $\{B_k\}_{k=1}^{\infty}$ be a smooth exhaustion of M, i.e. a sequence of precompact nonempty open subsets in M such that $\overline{B_k} \subset B_{k+1}$, $M = \bigcup_{k=1}^{\infty} B_k$.

Definition 1 [1]. Say that the functions $f_1(x)$ and $f_2(x)$ *are equivalent on* M and and write $f_1 \overset{M}{\sim} f_2$, if for some exhaustion $\{B_k\}_{k=1}^{\infty}$ of M we have

$$\lim_{k\to\infty} \|f_1(x) - f_2(x)\|_{C^0(M\setminus B_k)} = 0,$$

where $\|f(x)\|_{C^0(G)} = \sup\limits_{G} |f(x)|$.

Definition 2 [1]. Call the manifold M *L-precise* if on M there is a solution u of (1) such that $u \overset{M}{\sim} 1$.

Theorem 1. *Let M be L-precise, f be some bounded continuous function on M, and there are bounded solutions v and w of the equations $Lv = g(x)$ and $\Delta w = g(x)$, respectively on M, such that $v \overset{M}{\sim} f$ and $w \overset{M}{\sim} f$. Then there exists a solution u_1 of the equation $L_1 u = g(x)$ on M such that $u_1 \overset{M}{\sim} f$.*

Theorem 2. *Let f be some bounded continuous function on M, and there are bounded solutions v and w of the equations $Lv = g_1(x)$ and $L\omega = g_2(x)$, respectively on M, such that $v \overset{M}{\sim} f$ and $w \overset{M}{\sim} f$. Then there exists a bounded solution u of the equation $Lu = g(x)$ on M such that $u \overset{M}{\sim} f$.*

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MSC Primary 26D10, 26D15; Secondary 30C45.

Hardy's inequalities for the Jacobi weight and the Nehari-Pokornii type univalence conditions [1]

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Suppose that $\rho \in (0,1)$, $q > 0$ and $\nu \in \left[0, \frac{1}{q}\right]$. By j_ν we denote the first positive zero of the Bessel function J_ν of order ν. We prove that for any absolutely continuous function u such that $u(-\rho) = u(\rho) = 0$ and $u' \in L^2(-\rho, \rho)$ the following Hardy inequality for the Jacobi weight

$$P_q \int\limits_{-\rho}^{\rho} \frac{|g(t)|^2}{(1-t^2)^{2-q}} dt < \int\limits_{-\rho}^{\rho} g'^2(t) dt$$

is valid, where $q_0 \approx \frac{\pi^2}{18}$, $\alpha \in (0, q_0)$,

$$P_q = \begin{cases} 1 & , \quad \text{при} \quad q = 0; \\ \lambda_q & , \quad \text{при} \quad q \in (0, q_0); \\ \left(\frac{\lambda_\alpha}{2^\alpha}\right)^{\frac{1-q}{1-\alpha}} 2^q & , \quad \text{при} \quad q \in (q_0, 1]; \\ 2 & , \quad \text{при} \quad q = 1; \end{cases}$$

and $\sqrt{\lambda_q}/q$ is the first positive root of the following equation

$$-q^2\lambda^2 + q\lambda \frac{J_{\nu-1}(\lambda)}{J_\nu(\lambda)} = 0, \quad \lambda \in (0, j_\nu).$$

Using this inequality, we obtain Nehari-Pokornii type univalence conditions (see for example [1], [2]) for analytic in the unite disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ functions. Namely, we prove that

Theorem 1. *Suppose that f is meromorphic in $\mathbb{D}$ function. If $n \in \mathbb{N}$, a_k and μ_k, $k = \overline{1,n}$, are positive real numbers and*

$$|S_f(z)| \le \sum_{k=1}^{n} \frac{b_k A(\mu_k)}{(1-|z|^2)^{\mu_k}}, \quad z \in \mathbb{D},$$

where $b_k = \frac{2P_{2-\mu_k}}{A(\mu_k)} a_k$, $a_1 + a_2 + \ldots + a_n \le 1$, $0 \le \mu_1 \le \mu_2 \le \ldots \le \mu_n \le 2$ and

$$A(\mu) = \begin{cases} 2^{3\mu-1}\pi^{2(1-\mu)}, & 0 \le \mu \le 1, \\ 2^{3-\mu}, & 1 \le \mu \le 2; \end{cases}$$

Then the function f is univalent in $\mathbb{D}$.

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MSC 35B45

Uniform convergence of piece - wise polynomial solutions of minimal surface equation

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Let $\Omega \subset R^2$ be polygon bounded area and $f \in C^{m+1}(\overline{\Omega})$ be a solution of equation

$$\sum_{i=0}^{n} \frac{\partial}{\partial x_i} \left(\frac{f_{x_i}}{\sqrt{1+|\nabla f|^2}} \right) = 0. \tag{1}$$

Let φ be smooth enough function in Ω and continuous on $\partial\Omega$ such as $\varphi|_{\partial\Omega} = f|_{\partial\Omega}$. Consider triangulation of Ω, we will mark triangles of this triangulation as $T_1; T_2; ...; T_N$, sides of this triangles as G_s and $h = \max\limits_{1\le k\le N} \mathrm{diamT_k}$. We will mark the set of piece-wise polynomial functions u with degree m as P_m. If minimum of integral

$$I(\varphi+u) = \sum_{k=1}^{N} \int\limits_{T_k} \sqrt{1+|\nabla\varphi+\nabla u|^2} dx$$

reaches at the function $u^* \in P_m$ then we will call the function $f^* = \varphi + u^*$ piece-wise polynomial solution of (1).

As well we suppose that the triangulation has following property: there is a constant C which does not depend on h and the next inequality holds

$$h \cdot \sum_{innerG_s} \le C.$$

Theorem 1. *$f^* \rightrightarrows f$ on Ω when $h \to 0$.*

This result was obtained in [1] by using estimation of area functional

$$I(f) = \iint\limits_{\Omega} \sqrt{1+f_{x_1}^2+f_{x_2}^2} dx_1\, dx_2.$$

And the estimation is $|I(f^*) - I(f)| \le C^* K M h^{m+1}$, where $|\frac{\partial^{m+1} f}{\partial x_1^q \partial x_2^{m+1-q}}| \le M$, $q = 0, ..., m+1$, and constants K and C^* do not depend on $\{T_k\}_{k=1}^N$, Ω and f, m is a degree of approximation functions.

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MSC 53A10, 30C70, 31A15

Stability of the extreme surface given by the graph of the function[1]

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The work investigates the potential energy functional

$$W(\mathcal{M}) = \int\limits_{\mathcal{M}} \Phi(\xi)\, d\mathcal{M} + \int\limits_{\Omega} \Psi(x)\, dx. \qquad (1)$$

This is the sum of the area type functional and the bulk force density functional. Here ξ is a field of unit normals to the C^2-smooth surface $\mathcal{M}$, while $\Omega \in \mathbb{R}^{n+1}$ is some domain such that $\mathcal{M} \subset \partial\Omega$, and $\Phi, \Psi : \mathbb{R}^{n+1} \to \mathbb{R}$ are C^2-smooth functions. If the functions $\Phi(\xi) = 1, \Psi(x) = 0$, then the area-type functional and its extremals are minimal surfaces. If the functions $\Phi(\xi) = \text{const}, \Psi(x) = 1$, then surfaces of constant mean curvature are extremals. For area-type functional J. Simons [1] proved that the minimal surfaces given by the graphs of functions are stable and the Jacobi equation holds for them. In work [2] V.A. Klyachin showed that the Jacobi inequality holds for surfaces of constant mean curvature. Moreover, for the surfaces given by the graphs of functions, equality is achieved. Formulas for the first and second variations of functional (1) were obtained in papers [3]-[4]. They imply Theorem 1 proved in [5]. Applying Theorem 1 for surfaces defined by graphs of functions, and the equation of extremals, we obtain Theorem 2 on the stability. $G = \{G_{ij}\}_{i,j=1}^{n+1}$, $G_{ij} = \frac{\partial^2\Phi}{\partial\xi_i\partial\xi_j} + \delta_{ij}(\Phi - \langle D\Phi, \xi\rangle)$, $D\Phi = \left(\dfrac{\partial\Phi}{\partial\xi_1}, \dfrac{\partial\Phi}{\partial\xi_2}, \ldots, \dfrac{\partial\Phi}{\partial\xi_{n+1}}\right)$, δ_{ij} is the Kronecker symbol. $\mu_Q(\mathcal{M}) = \inf\left(\int\limits_{\mathcal{M}} G(\nabla h, \nabla h)\, d\mathcal{M} / \int\limits_{\mathcal{M}} Q h^2\, d\mathcal{M}\right)$, where $Q = \sum\limits_{i=1}^{n} k_i^2 G(E_i, E_i) - \left\langle \overline{\nabla}\Psi,\, \xi\right\rangle$, and the infimum is taken over all Lipschitz functions $h(m) : M \to \mathbb{R}$ satisfying the condition $\int\limits_{\mathcal{M}} h(m)\, d\mathcal{M} = 0$.

Theorem 1. *If there is a positive C^2-function $u(x) : \mathcal{M} \to \mathbb{R}$, satisfying $\triangle u(x) \leq -Qu(x)/\lambda$ in the metric of the surface $\mathcal{M}$, then $\mu_Q(\mathcal{M}) \geq 1$ and the surface $\mathcal{M}$ is stable.*

Theorem 2. *The extreme surface given by the function graph is stable.*

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MSC 31B99

Convexity of the p-harmonic radius of a circular sector[1]

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The concept of the inner (confomal) radius plays an important role in the geometric theory of functions. For a simply connected plane domain D of hyperbolic type, the conformal radius $R(D, a)$ at the point $a \in D$ is defined as the modulus of the derivative at zero of the conformal map of the unit disc onto D, which takes zero to a. In a more general situation, the definition of the conformal radius is given in terms of the Green's function. In the proof of Theorem 2 in [1], when searching for the maximum of the discrete Green's energy of a ring, the property of logarithmic convexity of the conformal radius of the annular sector (considered as a function of the sector angle) plays an important role. Namely, for fixed point $a > 0$ we denote by $R(\varphi)$ the conformal radius of the annular sector $\{z = re^{i\theta} : t < r < T, |\theta| < \pi\varphi\}$, $0 < \varphi \leq 1$, $0 \leq t < T \leq \infty$, calculated at the point a. In [1] it was proved that

$$\frac{1}{n}\sum_{k=1}^{n} \log R(\varphi_k) \leq \log R\left(\frac{1}{n}\sum_{k=1}^{n}\varphi_k\right) \tag{1}$$

for $0 < \varphi_1 < \ldots < \varphi_n \leq 1$. To prove (1), author used suitable conformal mappings, a radial averaging transformation and some geometric considerations.

A natural extension of the concept of conformal radius from a plane to Euclidean space $\mathbb{R}^d$ is the p-harmonic radius introduced in [2]. The most studied at the moment are the cases of harmonic ($p = 2$) and conformal ($p = d$) radii. The question arises about the validity of the inequality (1) in Euclidean space when the conformal radius is replaced by the p-harmonic radius. The above logic of the proof (1) fails in Euclidean space due to the boundedness of conformal mappings by Möbius mappings. In this talk, we show that the analogue (1) is also valid in Euclidean space. For the proof, a certain transformation of the moduli of families of curves is constructed. As a consequence, this transformation also implies the solution of one particular case of open problem 3 from monograph [3] (problem of A.Yu. Solynin).

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MSC 46E10

About one operator of the shift type

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Let $\pi(z)$ be an arbitrary polynomial. It is known that for any entire function $g(z)$ there is a unique π-symmetric representation

$$g(z) = \sum_{p=0}^{q-1} z^p g_p \circ \pi(z), \tag{1}$$

where $g_p(z)$ are entire functions. Let $\{\pi_0(z), ..., \pi_{q-1}(z)\}$ be an arbitrary collection of polynomials, not all of which are identically zero. We assume that the degree $q_p := \deg \pi_p(z)$ of the polynomial $\pi_p(z)$ does not exceed p. Consider a linear operator A acting from the space $O(\mathbf{C})$ to the space $O(\mathbf{C})$ by the rule

$$A : g(z) \mapsto \sum_{p=0}^{q-1} \pi_p\,(z)\, g_p \circ \pi(z),$$

where $g_p \circ \pi(z)$ is a π-symmetric coeficient of the representation (1).

Let Ω_0, Ω be convex domains in the complex plane $\mathbf{C}$, $\varepsilon > 0$, U_ε be the disc $\{h : |h| < \varepsilon\}$. We will assume that $\Omega_0 + U_\varepsilon \subseteq \Omega$ and the spaces of holomorphic functions $O(\Omega_0)$, $O(U_\varepsilon)$, $O(\Omega_0)$ and $O(\mathbf{C})$ are endowed with the topologies of uniform convergence on compact sets. Choose an arbitrary $h \in U_\varepsilon$ and consider a linear differential operator with constant coefficients

$$AT_h : f(z) \mapsto \sum_{n=0}^{\infty} \frac{A(e^{hz})^{(n)}(0)}{n!}\,(D^n f)\,(\zeta). \tag{2}$$

For any $f \in O(\Omega)$ the series (2) converges uniformly in (ζ, h) on compacts from the bicylinder $\Omega_0 \times U_{\varepsilon'}$ for any choice of convex domains Ω_0 and Ω satisfying the condition $\Omega_0 + U_{\varepsilon'} \subseteq \Omega$. This means that the image $AT_h(f)$ is an analytic function in the variable (ζ, h) on the bicylinder $\Omega_0 \times U_\varepsilon$ for any $f \in O(\Omega)$. This implies that the linear operator AT_h acts from the space $O(\Omega)$ to the space $O(\Omega_0)$. The operator $AT_h : O(\Omega) \to O(\Omega_0)$, defined by the rule (2) we call the π-shift operator (by step h).

The π-shift operator defined in this way develops all the existing specific procedures for generalizing the classical shift operator $f(z) \to f(z + h)$. Here we can talk only about two procedures that were previously used as the basis for the definition of the π -convolution operator and the π-convolution operator. The first of these procedures is more general. It reduces to the procedure described above if we put

$$\pi_0(z) := \operatorname{sym} z^0 \equiv s_0, ..., \pi_{q-1}(z) := \operatorname{sym} z^{q-1} \equiv s_{q-1}.$$

The second procedure is obtained by putting

$$\pi_0(z) = a_0 z^0, ..., \pi_{q-1}(z) = a_{q-1} z^{q-1},$$

where not all coefficients $a_0, ..., a_{q-1} \in \mathbf{C}$ are equal to zero. The talk will describe other properties of the operator (2).

MSC 30E25

On a linear conjugation problem, arising in the models of the flow over an obstacle.

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In hydrodynamics, in some models of flow over the obstacle, the following linear conjugation (Riemann) problem arises:

$$(\nu(\omega-\chi)^2\cosh(\chi)-\chi\sinh(\chi))V^+(\chi,\omega) = -(\nu(\omega+\chi)^2\cosh(\chi)-\chi\sinh(\chi))V^-(\chi,\omega)-$$
$$-\chi\sinh(\chi)V^+(\omega,\omega)-\frac{\sinh(\chi)(\omega^2-\chi^2)}{\chi}\left(\frac{1}{2\pi^2}+h(\omega)\right)+\cosh(\chi)(\omega^2-\chi^2)h(\omega),\quad (1)$$
$$\chi\in\mathbb{R},$$

with the symmetry condition

$$V^+(\xi,\omega) = V^-(-\xi,\omega),\quad \xi\in\mathbb{C}, \tag{2}$$

and asymptotic condition at the infinity

$$V^{\pm}(\xi,\omega) = -\frac{h(\omega)}{2\nu}+O\left(\frac{1}{\xi}\right),\quad \xi\to\infty,\quad \xi\in\mathbb{C}. \tag{3}$$

Here the functions $V^{\pm}(\chi,\omega)$ are the boundary values of analytic with respect to ξ at $\pm\mathrm{Im}\,\xi>0$ unknown functions $V^{\pm}(\xi,\omega)$, the function $h(\omega)$ is a sought one too. The peculiarities of the problem are: first, the right hand side of (1) contains the additional unknown $h(\omega)$, and second, the right hand side contains the value of unknown $V^+(\omega,\omega)$.

The variables (ξ,ω) are the spectral parameters in plane of the Fourier transform and ξ corresponds to the one space variable and ω corresponds to the time. In the problem (1–3), the variable ω plays the role of additional complex parameter, $\mathrm{Im}\,\omega\geqslant 0$.

We investigate the problem by the classical method of Gakhov, see e.g. [1]. It turns out, that at $\mathrm{Im}\,\omega>0$ the coefficient of the linear conjugation problem (1) $G(\chi,\omega) = (\nu(\omega+\chi)^2\cosh(\chi)-\chi\sinh(\chi))/(\nu(\omega-\chi)^2\cosh(\chi)-\chi\sinh(\chi))$ is continues and doesn't vanish and its index is equal to -2. So, when we construct the solution as the linear operator with respect to the right hand side in (1), in particular with respect to the $h(\omega)$ and $V^+(\omega,\omega)$, then two solvability conditions arise. One of them holds automatically because of the symmetry of the coefficient G and the right hand side of (1) with respect to χ, and the second solvability condition, together with the condition of the coincidence of the calculated value $V^+(\omega,\omega)$ of the solution (1) with the given in the right hand side, give the system of linear equations, allowing to determine $h(\omega)$ and $V^+(\omega,\omega)$ and so to construct the solution (1–3) completely.

When the parameter ω becomes real, the coefficient $G(\chi,\omega)$ degenerates (vanishes or approaches infinity). However, we show, that the limit value of the solution $V^{\pm}(\xi,\omega)$, when ω tends to the real value from the upper half plane, is continues, except the only point $\omega=0$. We find the asymptotic of the solution at $\omega\to 0$, what allows to describe the asymptotic of the unknowns in initial (physical) variables, when the time tends to infinity.

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On greedy summations of tensor products of operators

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We will call an *operator array* a family of bounded linear operators $\{U_s \colon H_1 \to H_2\}_{s\in S}$ between complex or real Banach spaces (S be an arbitrary set). Given an operator array $\{U_s\}_{s\in S}$ and a positive number ε, we refer to the array $\{U_s \mid \|U_s\| > \varepsilon\}$ as the ε-cut of the array. If all ε-cuts of operator array are finite and its sums strongly (i.e. with respect to the norm) converge, then the limit is called the greedy sum of the array and denoted

$$\sum_{s\in S} U_s = \lim_{\varepsilon\to 0} \sum_{\|U_s\|>\varepsilon} U_s\,.$$

Theorem 1. *Let $\{U_s \colon B_1 \to B_2\}_{s\in S}$ and $\{V_t \colon C_1 \to C_2\}_{t\in T}$ be greedy summable arrays, such that its tensor product $\{U_s \otimes V_t\}_{(s,t)\in S\times T}$ is also greedy summable. Then one has the equality*

$$\sum_{(s,t)\in S\times T} U_s \otimes V_t = \sum_{s\in S} U_s \otimes \sum_{t\in T} U_t\,.$$

The proof is based on the following identity for generating zeta-functions of operators arrays

$$\sum_{s\in S} U_s\|U_s\|^z \otimes \sum_{t\in T} V_t\|V_t\|^z = \sum_{(s,t)\in S\times T} (U_s \otimes V_t)\|U_s \otimes V_t\|^z\,.$$

The proof, being similar to the one given in the paper [1] for the greedy multiplication theorem, is based on the theory of operator-valued function of complex variable. Let us note the theorem 1 in finite dimensional case does not follow from the corresponding matrix multiplication theorem from the paper [1]. These theorems deal with different types of multiplication and greed.

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Free Boundary Value Problems and Symmetrization of the Functions and Domains

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The efficiency of symmetrization method is well known [1]. We propose here some additional applications of it to the different variational problems arising in the study of Free Boundary Value Problems. First of them deals with the generalization of the Free Boundary Value Problem corresponding to the axisymmetric liquid's flow of Ryaboushinsky type past the body of the meridional section B with additional cavern of the meridional section W [2]. Let Ψ be the function of the flow defined in its meridional section and $\sigma := \partial W \setminus B$. Then instead of classical condition $\frac{1}{2y^2}\left|\Psi\right|^2 = \lambda$ on σ the condition $\frac{1}{2y^2}\left|\Psi\right|^2 + \kappa H + \theta K = \lambda$ is considered, the coordinate y being orthogonal to the flow direction along the axis x. The Free Boundary Value Problem for the function Ψ in the unbounded domain $\mathbb{R}^2 \setminus (B \cup W)$ reduces to the variational problem for the variational problem for the functional $F(\Psi, \sigma)$, $F(\Psi, \sigma) := M(\Psi, \sigma) + (1 - 2\lambda) \cdot V(\sigma) + \kappa \cdot S(\sigma) + 2\theta \cdot \Xi(\sigma)$. Here H, K are mean and Gaussian curvatures of the surface representing part of cavern's boundary dividing it from the liquid's flow, κ, θ, λ being the constants determined by the liquid's properties, $M(\Psi, \sigma)$ is the virtual mass of the liquid's flow, $V(\sigma)$ – cavern's volume, $S(\sigma)$ – area of the surface representing part of cavern's boundary dividing it from the liquid's flow and $\Xi(\sigma)$ – the functional whose first variation is determined by the Gaussian curvature of it. The functions Ψ have zero values on the boundary of $B \cup W$, bounded virtual mass and asymptotic behavior $\Psi(x, y) \sim \frac{y^2}{2}$ at infinity. The symmetrization method applied to the domains $\mathbb{R}^2 \setminus (B \cup W)$ and to the functions Ψ permits us to substitute the rectifiable curves σ_n of the initial arbitrary minimizing sequence by the curves σ_n^*, monotone in each quadrant. This leads to the compactness of the sequence $\{\sigma_n^*\}$ and as the result to the compactness of the sequence $\{(\Psi_n^*, \sigma_n^*)\}$ with flow's functions Ψ_n^* defined in the domains corresponding to the curves σ_n^*. In the study of uniqueness of the variational solution of Free Boundary Value Problem the Steiner symmetrization of the caverns obtained by homotopic transformation determined by two possible solutions and combined with their Swartz symmetrization help to prove the uniqueness of the solution of the problem described [3]. The method of Steiner symmetrization of domains and functions proves to be useful also in the study of generalization of stationary Stefan problem of growing crystals [4]. This time it is applied to the functions satisfying mixed boundary condition instead of the Dirichlet condition as in the previous case.

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Search for an equilibrium nano-drop. Extended Young's formula

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To study the equilibrium drops of small diameter, it is necessary to consider a model that takes into account the thickness of the intermediate layer l_p.

The pressure difference $P_1 - P_2$ at each point of the equilibrium drop surface depends on the mean curvature $2H$ and the Gaussian curvature K,

$$P_1 - P_2 = \sigma_{LG} \left[2H + K l_p \right] = g\rho x(s) + \lambda. \tag{1}$$

Here
λ – Langrange multiplier,
g – acceleration of gravity,
ρ – density of the liquid.

To obtain an equilibrium drop the variation method is used. As the intermediate layer is taken into account, it is necessary to include the intermediate layer formation's energy functional $E^M_{\text{int.l.}}$ into that of the total energy

$$E^M_{\text{int.l.}}(S) = 2\pi \int_0^L f^M\left(\dot{x}(s)\right) ds,$$

s – natural parameter of the generating line defined parametrically by $(x(s), y(s))$,

$$f^M(\chi) = \sqrt{1-\chi^2} \left(\frac{-1}{2} \arcsin \chi + D \right) \; \forall \, D \in \mathbb{R}.$$

As a result, the axisymmetric surface satisfying (1) is obtained. This satisfies the following generalized contact angle condition

$$\cos\theta = \frac{(\sigma_{SG} - \sigma_{SL})}{\sigma_{LG}} + \frac{l_p}{4Y_A} \left[2\theta + \pi + \sin 2\theta \right], \tag{2}$$

θ – drop wetting angle,
σ_{LG}, σ_{SG}, σ_{SL} — surface tension coefficients between liquid and gas, solid and gas, solid and liquid, respectively,
Y_A – the radius of the wetting spot.

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MSC 46E10

On the definition of a convolution-type operator in the space of analytic functions

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For the theory of spectral synthesis in spaces of analytic functions, continuous endomorphisms $A : O(\mathbf{C}) \to O(\mathbf{C})$ are of particular importance if the images of monomials $A(\xi^n)$ have minimal type for order 1. In such a situation, the following inequality becomes of key importance

$$\overline{\lim_{n\to\infty,\lambda\to\infty}} \frac{1}{n}\left(\frac{|A(\xi^n)(\lambda)|}{\exp\varepsilon|\lambda|}\right)^{\frac{1}{n}} \le \frac{1}{\varepsilon e} \tag{1}$$

Suppose $\varepsilon' > 0$ and Ω_0, Ω are simply connected domains in $\mathbf{C}$ satisfying the condition $\Omega_0 + U_{\varepsilon'} \subseteq \Omega$. The linear differential operator with characteristic function $A(e^{h\xi})$, which we denote by the symbol AT_h is called *A-shift operator*, if for any $h \in U_{\varepsilon'}$ it takes $O(\Omega)$ to $O(\Omega_0)$ and is continuous. Choose an arbitrary A-shift operator AT_h, arbitrary $f \in O(\Omega)$, $\varepsilon \in (0, \varepsilon')$ and a functional $S \in O^*(\Omega_0)$. The function $\langle S, AT_h(f)\rangle$ is denoted on the disc $U_{\varepsilon'}$. Moreover, a linear operator AM_S, which assigns $\langle S, AT_h(f)\rangle$ to any function $f \in O(\Omega)$, is defined on the space $O(\Omega)$. If the linear operator AM_S acts from the space $O(\Omega)$ into the space $O(U_\varepsilon)$ and is continuous, then it is called *the A-convolution operator*.

It is said that for the homogeneous A-convolution equation

$$AM_S(f) = 0, \quad f \in O(\Omega),$$

the approximation theorem holds if any solution of this equation can be approximated in the space $O(\Omega)$ with its exponential solutions. It is known that the approximation theorem for the homogeneous A-convolution equation is true if Ω is a convex domain and the following conditions are satisfied:

1) for any $\varepsilon > 0$ the inequality (1) holds;

2) the continuous endomorphism $A : O(\mathbf{C}) \to O(\mathbf{C})$ is a π-symmetrization operator, that is

$$A(1) = 1, \quad A(\mathbf{C}[\xi]) = \mathbf{C}[\pi(z)],$$

where $\mathbf{C}[\lambda]$ is the ring of polynomials in λ;

3) the class of all entire π-symmetric functions is factorizable with the following relation

$$|\ln|g_1 \circ \pi(z)| - \ln|g_2 \circ \pi(z)|| = o(|z|), \quad z \to \infty, \quad z \notin E_0.$$

Condition 1) turns out to be redundant. More precisely, the following theorem is true.

Theorem 1. *If conditions 2) and 3) are satisfied, then the approximation theorem for homogeneous A-convolution equation is valid for any choice of convex domains Ω_0 and Ω, satisfying the condition $\Omega_0 + U_{\varepsilon'} \subseteq \Omega$, and the functional $S \in O^*(\Omega_0)$ iff the inequality (1) holds for all $\varepsilon \in (0, \varepsilon')$.*

This theorem essentially extends the family of homogeneous equations of convolution type in spaces of analytic functions, which have an exhaustive solution description.

MSC 46T12, 46F05

Iterated ultra-hyperbolic equation with Bessel operator

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Let $n=p+q$, p and q are natural, $\gamma{=}(\gamma',\gamma'')$, $\gamma'{=}(\gamma_1,...,\gamma_p)$, $\gamma''{=}(\gamma_{p+1},...,\gamma_{p+q})$, $\gamma_i>0$, $i{=}1,...,n$, $\mathbb{R}^n_+{=}\{x{=}(x_1,\ldots,x_n)\in\mathbb{R}^n,\ \ x_1{>}0,\ldots,x_n{>}0\}$, $x'\in\mathbb{R}^p_+$, $x''\in\mathbb{R}^q_+$, $x{=}(x',x''){\in}\mathbb{R}^n_+$, $\mathbb{R}^n_+{=}\mathbb{R}^p_+{\times}\mathbb{R}^q_+$.

B-ultra-hyperbolic equation or singular ultra-hyperbolic equation has the form

$$\Box_\gamma u=0,\quad u=u(x),$$

where $\Box_\gamma$ is homogeneous linear differential operator of the form

$$\Box_\gamma=(\Delta_{\gamma'})_{x'}-(\Delta_{\gamma''})_{x''}=B_{\gamma_1}+...+B_{\gamma_p}-B_{\gamma_{p+1}}-...-B_{\gamma_{p+q}},$$

$$(\Delta_{\gamma'})_{x'}=\sum_{i=1}^{p}(B_{\gamma_i})_{x_i},\ \ (\Delta_{\gamma''})_{x''}=\sum_{j=p+1}^{p+q}(B_{\gamma_j})_{x_j},\ \ B_{\gamma_i}{=}\frac{\partial^2}{\partial x_i^2}{+}\frac{\gamma_i}{x_i}\frac{\partial}{\partial x_i},\ \ i=1,...,n.$$

Iterated B-ultra-hyperbolic equation we will call the equation of the form

$$\Box_\gamma^k u=f, \tag{1}$$

where $k{\in}\mathbb{N}$, $f=f(x)$ is a suitable function.

In this report we find the fundamental solution to the equation $\Box_\gamma^k u=f$ using the results obtained for weighted generalized functions.

Let $x\in\mathbb{R}^n_+$, $n=p+q$, $p,q\in\mathbb{N}$. Fundamental solution to the equation (1) is weighted generalized function u such that

$$\Box_\gamma^k u=\delta_\gamma, \tag{2}$$

where δ_γ is weighted delta-function (see [1], p. 18).

Theorem 1. *Except when $n+|\gamma|=2,4,6,...$ and $k\geq\frac{n+|\gamma|}{2}$ weighted generalized function*

$$u=(-1)^k\frac{e^{\pm i\frac{\pi(q+|\gamma''|)}{2}}\Gamma\left(\frac{n+|\gamma|}{2}-k\right)}{4^k(k-1)!|S_1^+(n)|_\gamma\Gamma\left(\frac{n+|\gamma|}{2}-1\right)}(P\pm i0)_\gamma^{-\frac{n+|\gamma|}{2}+k}$$

is the fundamental solution to the equation $\Box_\gamma^k u=f$ in the sense (2).

If $n{+}|\gamma|{=}2,4,6,...$ and $k{\geq}\frac{n+|\gamma|}{2}$ then weighted generalized function

$$(P+i0)_\gamma^{-\frac{n+|\gamma|}{2}+k}{=}(P-i0)_\gamma^{-\frac{n+|\gamma|}{2}+k}$$

is a solution to a homogeneous equation $\Box_\gamma^k u=0$.

For the definitions of weighted generalized functions and $(P{\pm}i0)_\gamma^\lambda$ see [1], p. 163, 188.

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MSC 46E35, 31C45

Removable sets for weighted Sobolev spaces with a Muckenhoupt weight

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In [1] Vodop'yanov and Gol'dstein gave a criterion of removable singularities for Sobolev spaces $L^1_p(\Omega)$, $W^1_p(\Omega)$ in terms of NC_p-sets, $1 < p < \infty$. An NC_p-set can be considered as a p-analog of an NED-sets, earlier introduced by Väisälä [2] as a result of generalizing the concept of NED-set in R^2 [3] to R^n, $n \geq 2$. Also note that the definitions of NC_p-set in open set $\Omega \subset R^n$ or an NED-set in R^n are based on condensers whose plates are the pair of arbitrary disjoint continuums located outside this set in Ω or in R^n, respectively. Latter [4] Dymchenko and Shlyk obtained similar assertions about removable singularities for the spaces $L^1_{p,w}(\Omega)$, $W^1_{p,w}(\Omega)$ in terms of $NC_{p,w}$-sets in Ω (as well as the initial definition of NED-set by Ahlfors-Beurling [3] in R^2) is based on condensers formed by an arbitrary coordinate rectangles Π, $\bar{\Pi} \subset \Omega$, and by any of its opposite facets as plates.

In the talk we define $NC_{1,w}$-set in Ω with Muckenhoupt A_1-weight w and give criteria of equalities $L^1_{1,w}(\Omega \setminus E) = L^1_{1,w}(\Omega)$, $W^1_{1,w}(\Omega \setminus E) = W^1_{1,w}(\Omega)$ in terms of E as an $NC_{1,w}$-set, $NED_{1,w}$-set or as a set with the $(1, w)$-girth condition. This presentation is based on a work [5].

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Maximum principle extension for quasilinear parabolic equations.

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We discuss versions of the maximum principle for quasilinear parabolic equations for an unknown vector valued function $u = u(t, x_1, \dots, x_n) \in \mathbb{R}^m$, in the simplest case having the form $u_t + (f(u) \cdot \nabla)u = \nu \Delta u$. Here $f : \mathbb{R}^m \to \mathbb{R}^n$ is a general nonlinearity. The positive parameter ν has the physical meaning of viscosity. Most common, the maximum principle is formulated for the case of bounded cylindrical domains $\Pi = [0, T) \times \Omega$, where the function u is assumed to be continuous up to the boundary:

$$\sup_{\Pi} |u| \leqslant \max\{M_0, M_\Gamma\}, \quad \text{where } M_0 = \sup_{\{0\}\times\Omega} |u| \text{ and } M_\Gamma = \sup_{[0,T)\times\partial\Omega} |u|.$$

This statement can be trivially extended to the case $\Omega = \mathbb{R}^n$, if M_Γ is understood as a corresponding upper limit. Here $|u|$ stands for Euclidean norm of $u \in \mathbb{R}^m$.

In [8] for $m = 1$ it is, shown that $\sup\limits_{[0,T)\times\mathbb{R}^n} u \leqslant M_0$ if $M_\Gamma < +\infty$.

The purpose of this work is to relax the last condition. We show the finiteness of M_Γ condition can be replaced by a condition of the form $|u(t, x)| \leqslant C(1 + |x|^\alpha)$ for some $\alpha < 2$ depending on the nonlinearity of f. Such results are important in the study of various evolutionary equations of parabolic type. In particular, when we constructing a priori estimates for various norms of solutions for families of parabolic equations [4,5]. This becomes especially relevant when other explicit methods (for example, [3]) cannot be applied. Along with the lower bounds [2], the new a priori estimates can be useful in the mathematical description of the turbulence phenomena [4,6] ($\nu \to 0$).

Analyzing various special cases of the function f, in particular the following examples of integrable equations $u_t - u_{xx} = 0$, $u_t \pm uu_x - u_{xx} = 0$, it can be hypothesized that the growth condition at infinity can be significantly relaxed. Nevertheless, it can be shown that it is impossible to completely abandon growth restrictions in the general case [1,7].

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MSC 62M15, 46L60, 47L90, 70H06, 70F05

Structure of Essential Spectra and Discrete Spectrum of Four-Electron Systems in the Impurity Hubbard Model in a Quintet State

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The spectrum and wave functions of the system four electrons in a crystal described by the Hubbard Hamiltonian in the quintet and singlet states and triplet states were studied in [1,2]. Here, we consider the energy operator of four electron systems in the impurity Hubbard model and describe the structure of essential and discrete spectra of the system for quintet state. The Hamiltonian of the chosen model has the form $H = A\sum_{m,\gamma} a^+_{m,\gamma}a_{m,\gamma} + B\sum_{m,\tau,\gamma} a^+_{m,\gamma}a_{m+\tau,\gamma} + U\sum_m a^+_{m,\uparrow}a_{m,\uparrow}a^+_{m,\downarrow}a_{m,\downarrow} + (A_0 - A)\sum_\gamma a^+_{0,\gamma}a_{0,\gamma} + (B_0 - B)\sum_{\tau,\gamma}(a^+_{0,\gamma}a_{\tau,\gamma} + a^+_{\tau,\gamma}a_{0,\gamma}) + (U_0 - U)a^+_{0,\uparrow}a_{0,\uparrow}a^+_{0,\downarrow}a_{0,\downarrow}$. Here $A\,(A_0)$ is the electron energy at a regular (impurity) lattice site, $B\,(B_0)$ is the transfer integral between (between electron and impurities) neighboring sites (we assume that $B > 0\,(B_0 > 0)$ for convenience), and the summation over τ ranges the nearest neighbors, $U\,(U_0)$ is the parameter of the on-site Coulomb interaction of two electrons in the regular (impurity), γ is the spin index, and $a^+_{m,\gamma}$ and $a_{m,\gamma}$ are the respective electron creation and annihilation operators at a site $m \in Z^\nu$. The Hamiltonian H acts in the antisymmetric Fo'ck space $\mathcal{H}_{as}$. Let φ_0 by the vacuum vector in the space $\mathcal{H}_{as}$. The quintet state corresponds to the free motion of four electrons over the lattice with the basic functions $q^2_{m,n,k,l\in Z^\nu} = a^+_{m,\uparrow}a^+_{n,\uparrow}a^+_{k,\uparrow}a^+_{l,\uparrow}\varphi_0$. The subspace $\mathcal{H}^q_2$, corresponding to the quintet state is the set all vectors of the form $\psi^q_2 = \sum_{m,n,k,l\in Z^\nu} f(m,n,k,l)q^2_{m,n,k,l\in Z^\nu}$, $f \in l^{as}_2$, where l^{as}_2 is the subspace of antisymmetric functions in the space $l_2((Z^\nu)^4)$. Denote by H^q_2 the restriction of H to the subspace $\mathcal{H}^q_2$. We let $\varepsilon_1 = A_0 - A$, $\varepsilon_2 = B_0 - B$, and $\varepsilon_3 = U_0 - U$.

Theorem 1. *Let $\nu = 1$, and $\varepsilon_2 = -B$, and $\varepsilon_1 < -2B$ (respectively, $\varepsilon_2 = -B$, and $\varepsilon_1 > 2B$). Then the essential spectrum of the operator H^q_2 is consists of the union of four segments $\sigma_{ess}(H^q_2) = [4A - 8B, 4A + 8B] \cup [3A - 6B + z, 3A - 6B + z] \cup [2A - 4B + 2z, 2A + 4B + 2z] \cup [A - 2B + 3z, A + 2B + 3z]$ and discrete spectrum of the operator $H^q_2$is consists of a single eigenvalue, $\sigma_{disc}(H^q_2) = 4z$, where $z = A + \varepsilon_1$.*

Theorem 2. *Let $\nu = 1$, and $\varepsilon_2 > 0$, and $-\frac{2(\varepsilon_2^2+2B\varepsilon_2)}{B} < \varepsilon_1 < \frac{2(\varepsilon_2^2+2B\varepsilon_2)}{B}$, then the essential spectrum of the operator H^q_2 is consists of the union of the nine segment $\sigma_{ess}(H^q_2) = [4A - 8B, 4A + 8B] \cup [3A - 6B + z_1, 3A - 6B + z_1] \cup [3A - 6B + z_2, 3A - 6B + z_2] \cup [2A - 4B + 2z_1, 2A + 4B + 2z_1] \cup [2A - 4B + 2z_2, 2A + 4B + 2z_2] \cup [2A - 4B + z_1 + z_2, 2A + 4B + z_1 + z_2] \cup [A - 2B + 3z_1, A + 2B + 3z_1] \cup [A - 2B + 3z_2, A + 2B + 3z_2] \cup [A - 2B + 2z_1 + z_2, A + 2B + 2z_1 + z_2] \cup [A - 2B + z_1 + 2z_2, A + 2B + z_1 + 2z_2]$, and discrete spectrum of the operator H^q_2 is consists of five eigenvalues, $\sigma_{disc}(H^q_2) = \{4z_1, 4z_2, 3z_1 + z_2, z_1 + 3z_2, 2z_1 + 2z_2\}$.*

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MSC 46E10

On homogeneous q-sided convolution equations with periodic indicator

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Let U_ε be an open disc $|z| < \varepsilon$; Ω_0, Ω are convex domains in $\mathbf{C}$, $\Omega_0 + U_\varepsilon \subseteq \Omega$. We assume that the spaces of analytic functions $O(\Omega_0)$, $O(\Omega)$ are endowed with the usual topology. By the symbol $T_h^{(k)}$ we denote the shift operator $f(z) \mapsto f(z + \omega_q^k h)$, where $\omega_q = \exp \frac{2\pi i}{q}$. The operator

$$AT_h : f(z) \mapsto a_0 T_h^{(0)}(f) + ... + a_{q-1} T_h^{(q-1)}(f),$$

acting from the space $O(\Omega)$ to the space $O(\Omega_0)$ is usually called *the q-sided shift operator* (by step $h \in U_\varepsilon$). Here, not all complex $a_0, ..., a_{q-1}$ are equal to zero. The kernel of this operator coincides with the space of solutions of *the homogeneous q-sided convolution equation*

$$\langle S, AT_h(f) \rangle = 0, \quad f \in O(\Omega),$$

where S is a linear continuous functional on the space $O(\Omega_0)$. We say that an *approximation theorem* holds for this equation if any solution of this equation can be approximated in the space $O(\Omega)$ by equation elementary solutions (exponential polynomials) for any choice of the convex domain Ω and the functional S.

The q-sided shift operator AT_h coincides with the differential operator of infinite order

$$f(z) \to \sum_{n=0}^{\infty} b_n \frac{h^n}{n!} (D^n f)(z), \tag{1}$$

where

$$b_n := \sum_{k=0}^{q-1} a_k \omega_q^{kn}, \; n \in \{0, 1, ...\}$$

and the series (1) converges uniformly on compact sets from Ω_0. The coefficients b_n depend on n in a periodic manner, that is, for any $n \in \mathbf{Z}_+$ the equality $b_{n+q} = b_n$ holds. Hence, among the coefficients b_n, $n \in \{0, ..., q-1\}$ there are nonzero. These coefficients are listed by the *indicator*

$$n_A := \{n_1, ..., n_\nu\}$$

of the operator (1). The following theorem is true.

Theorem 1. *The approximation theorem for* the homogeneous q-sided convolution equation *is true if and only if the defining coefficients of this equation satisfy the following conditions:*

1) $q = q_0 \nu$ *for some* $q_0, \nu \in \{1, ..., q\}$;

2) $a_{n+\nu} = \omega_q^{-\nu n_1} a_n$ *for any* $n \in \{0, ..., q - \nu - 1\}$;

3) $\sum_{n=1}^{\nu} \omega_q^{(q_0(k-1)+n_1)n} a_n \neq 0$ *for all* $k \in \{1, ..., \nu\}$.

MSC 28A80

Finiteness properties for self-similar continua[1]

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Let $\mathcal{S} = \{S_1, ..., S_m\}$ be a system of injective contraction maps in $\mathbb{R}^n$. A non-empty compact set K satisfying the equation $K = \bigcup\limits_{i=1}^{m} S_i(K)$ is called *the attractor of the system* $\mathcal{S}$ and the sets $K_i = S_i(K)$, where $i = 1, ..., m$, are called *the pieces* of the set K. We consider the case when the maps S_i are the similarities and the attractor K is connected. In this case we say K is a *self-similar continuum.*

We say the self-similar continuum K (as well as the system $\mathcal{S}$) has finite intersection property if for any non-equal i, j, the intersection of pieces $K_i \cap K_j$ is finite and $\#(K_i \cap K_j) \le s$ for some s. In this case we say that $\mathcal{S}$ is a *FI(s)-system of contractions.* We define the intersection graph of the system $\mathcal{S}$ and prove the folowing theorem.

Theorem 1. *Let $\mathcal{S}$ be a system of injective contraction maps in a complete metric space X, which satisfies finite intersection property. The attractor K of the system $\mathcal{S}$ is a dendrite iff the intersection graph of the system $\mathcal{S}$ is a tree.*

If a FI(s)-system $\mathcal{S}$ of similarities in $\mathbb{R}^n$ satisfies open set condition [1], then we prove that the number of addresses, ramification number and topological order for each point $x \in K$ and for the boundary $\partial K_{\mathbf{j}}$ of each piece $K_{\mathbf{j}}$ of the attractor K have a uniform finite upper bound.

This allows us extend the parameter matching theorem for polygonal dendrites [2], to a general case of self-similar continua in $\mathbb{C}$, satisfying the finite intersection property.

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MSC 26B15

On the uniform convergence of polynomial solutions to the equilibrium capillary surface equation

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We consider the issue on convergence of approximate solutions for the equilibrium capillary surface equation

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\frac{f_{x_i}}{\sqrt{1+|\nabla f|^2}} \right) = \varkappa f \tag{1}$$

in domain Ω subject to the boundary condition

$$\left. \frac{\langle \nabla f, \nu \rangle}{\sqrt{1+|\nabla f|^2}} \right|_{\partial\Omega} = \varphi, \tag{2}$$

where $\varphi \in C^1(\overline{\Omega})$, $\varkappa = const > 0$, $\nu = (\nu_1, \ldots, \nu_n)$ is the unit outer normal at the points of the boundary $\partial\Omega$, where it exists. Previously we prove [2] the uniform convergence of polynomial solutions of the minimal surface equation in such domains. The uniform convergence of piecewise linear solutions to the exact solution of the equation of an equilibrium capillary surfaceis considered in the article [1]. Suppose that $\Omega \subset \mathbf{R}^n$ is a bounded domain. For a natural number N we denote by L_N the set all polynomials such that the degree in each variable does not exceed N. Assume that $\varphi \in C^1(\overline{\Omega})$. Consider the problem on finding a polynomial $v_N^* \in L_N$ on which the area integral

$$E(v_N) = \int\limits_{\Omega} \sqrt{1+|\nabla v_N|^2}\, dx + \frac{\varkappa}{2} \int\limits_{\Omega} v_N^2(x)\, dx - \int\limits_{\partial\Omega} v_N \varphi\, ds \to \min, \quad v_N \in L_N. \tag{3}$$

attains its minimum. Function $v_N^* \in L_N$ is called a polynomial solution to boundary value problem (1)–(2) if for each polynomial $v_N \in L_N$ identity holds true

$$\int\limits_{\Omega} \frac{\langle \nabla v_N^*, \nabla v_N \rangle}{\sqrt{1+|\nabla v_N^*|^2}}\, dx + \varkappa \int\limits_{\Omega} v_N^*(x) v_N(x)\, dx - \int\limits_{\partial\Omega} v_N \varphi\, ds = 0. \tag{4}$$

Note, that the conditions (3) and (4) are equivalent to.

Theorem. *Let f be a solution to the problem (1)–(2), $f, \varphi \in C^k(\overline{\Omega})$. Suppose that the domain Ω has a C^2 - smooth boundary. Then the sequence v_N^* converges uniformly to the solution f for a $N \to \infty$.*

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On some generalisations of the Hawaii conjecture

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In the talk, we will provide a progress on a generalisation of Hawaii conjecture stating that the number of the critical points of the logarithmic derivative of a real polynomial does not exceed the number of non-real roots of the polynomial.

There will be presented some new approaches promising to simplify the proof of the Hawaii conjecture, and the general situation with the Laguerre and Newton inequalities for real polynomials and their generalisations will be discussed.

This is a joint work with A. Vishnyakova and O. Katkova.

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УДК 517.55, 517.95

On conjugation problem in the theory of elliptic equations

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Пусть плоский сектор имеет вид $C^a_+ = \{x \in \mathbb{R}^2 : \ x_2 > a|x_1|, \ a > 0\}$, A – эллиптический псевдодифференциальный оператор с символом $A(\xi)$, удовлетворяющим условию

$$c_1 \le |A(\xi)(1+|\xi|)^{-\alpha}| \le c_2, \quad \alpha \in \mathbb{R}.$$

Рассматривается задача нахождения нетривиальной пары функций $u_+ \in H^{s_1}(C^a_+)$, $u_- \in H^{s_2}(\mathbb{R}^2 \setminus \overline{C^a_+})$ из соответствующих пространств Соболева–Слободецкого [1], удовлетворяющих следующим уравнениям

$$(Au_+)(x) = 0, \quad x \in C^a_+,$$

$$(Au_-)(x) = 0, \quad x \in \mathbb{R}^2 \setminus \overline{C^a_+},$$

и условий, при которых такая пара может быть определена единственным образом.

Применение методов и техники многомерного комплексного анализа позволяет в ряде случаев записать общие решения каждого из фигурирующих уравнений, однако при этом от символа $A(\xi)$ требуется наличие специальной волновой факторизации [1], а именно, возможность представления в виде

$$A(\xi) = A_{\neq}(\xi) \cdot A_{=}(\xi),$$

где сомножители допускают аналитическое продолжение в радиальные трубчатые области $T(\pm \overset{*}{C^a_+})$ над сопряженными конусами и удовлетворяют некоторым оценкам роста, порядок которого определялся индексом волновой факторизации æ $\in \mathbb{R}$. В зависимости от корреляции между æ и s картина разрешимости этих уравнений имела совершенно различный вид.

Мы исследуем случай æ $- s = 1 + \delta, |\delta| < 1/2$ и подбираем различные типы дополнительных условий на пары решений внутри и вне конуса, при которых такая пара может быть однозначно определена. В частности, одним из условий выбирается интегральное условие [2] на искомые функции, к которому добавляются линейные соотношения, связывающему граничные значения u_+, u_- на ∂C^a_+. Вопрос об однозначной разрешимости сформулированной задачи сопряжения сводится к однозначной разрешимости полученной системы линейных интегральных уравнений. Эта система строится по элементам волновой факторизации и коэффициентам линейного соотношения граничных значений $u_\pm$. При некоторых дополнительных предположениях о структуре символа $A(\xi)$ последняя система может быть сведена к системе линейных алгебраических уравнений.

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MSC 58J05

The Liuoville-type theorem for the elliptic inequality of a special type on Quasi-model Riemannian manifolds

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The paper is devoted to the problems of existence of positive solutions of the elliptic inequality

$$-\Delta u \geq u^q,\ q > 1 \tag{1}$$

on Quasi-model Riemannian manifolds. Let us describe them in more detail.

We consider the Riemannian manifold M that is isometric to the direct product $R_+ \times S_1 \times S_2 \times \dots \times S_k$ (where $R_+ = (0, +\infty)$ and S_i is a compact Riemannian manifolds without boundary) with the metric:

$$ds^2 = dr^2 + g_1^2(r)d\theta_1^2 + \dots + g_k^2(r)d\theta_k^2.$$

Here $g_i \in C^1(R_+)$ is a positive function on R_+. Suppose also that $n_i = dim\, S_i$.

As the solution of inequality (1) on M we consider a function $u \in C^1(M)$ such that for any set $\Omega \subset M$ and for any positive function $\phi(x) \in C_0^1(\Omega)$ satisfies the equation

$$\int\limits_{\Omega} \nabla u \cdot \nabla \phi \, dx \leq \int\limits_{\Omega} u^q \, \phi(x) \, dx,$$

where ∇u is the gradient of u. Then the following statement is true.

Theorem 1. *Let the manifold M be such that*

$$\limsup_{\rho\to\infty} \rho^{\frac{2}{q-1}} \left(\frac{\int\limits_{\rho/4}^{2\rho} G(r)dr}{\int\limits_{\rho/2}^{\rho} G(r)dr} \right)^{\frac{1}{q-1}} \int\limits_{2\rho}^{\infty} \frac{ds}{G(s)} = +\infty,$$

where $G(r) = g_1^{n_1}(r) \cdot g_2^{n_2}(r) \cdot \ldots \cdot g_k^{n_k}(r)$.

Then any non-negative solution of inequality (1) on M is the identical zero.

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On the Hadamard–Chen Fractional Integro-differentiation

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The fractional integro-differentiation by Hadamard-Chen

$$(\mathrm{J}_c^\alpha\varphi)(x) := \begin{cases} \frac{1}{\Gamma(\alpha)}\int_c^x \frac{\varphi(t)}{\left(\ln\frac{x}{t}\right)^{1-\alpha}}\,\frac{dt}{t}, x > c, \\ \frac{1}{\Gamma(\alpha)}\int_x^c \frac{\varphi(t)}{\left(\ln\frac{t}{x}\right)^{1-\alpha}}\,\frac{dt}{t}, x < c \end{cases}$$

and

$$(D_c{}^\alpha f)(x) = \frac{1}{\chi(\alpha,l)}\int_0^\infty \frac{(\tilde{\Delta}_t^l f_{c+})(x) + (\tilde{\Delta}_{t^{-1}}^l f_{c-})(x)}{|lnt|^{1+\alpha}}\frac{dt}{t}$$

respectively, where $\left(\tilde{\Delta}_{t^{\pm 1}}^l f\right)(x) = \sum_{k=0}^l (-1)^k \binom{l}{k} f\left(x\cdot t^{\pm k}\right)$, $l > \alpha > 0, c > 0, x \in \mathbb{R}_+$, $f_{c+}(x) = \begin{cases} f(x), & x > c \\ 0, & x < c, \end{cases}$ $f_{c-}(x) = \begin{cases} 0, & x > c \\ f(x), & x < c. \end{cases}$ Since the Chen construction can be applied to functions with an arbitrary growth when $x \to \infty$ or $x \to 0$, this construction is more convenient when applied to such functions than the integro-differentiation by Hadamard [1] itself. As usual, the fractional derivative is to be treated as a certain limit. To this end, several types of different "truncation" of the Hadamard-Chen fractional derivative are introduced, denoted by

$$(D_{c,\rho}^\alpha f)(x), (\dot{D}_{c,\rho}^\alpha f)(x), (D_{c,\tilde{\rho}}^\alpha f)(x),$$

where $\frac{1}{\sqrt[l]{e}} < \rho < 1, \tilde{\rho} = |ln\frac{x}{c}|ln\frac{1}{\rho}$.

Theorem 1. *Let* $f = J_c^\alpha\varphi, \varphi \in L^p(\mathbb{R}_+, \frac{dx}{x})$(*or* $\varphi \in L_{loc}^p(\mathbb{R}_+, \frac{dx}{x})), 1 \le p < \infty, l > \alpha > 0, c > 0$ *and* $0 < \rho < 1$. *Then*

$$\lim_{\rho\to 1-0}(D_{c,\rho}^\alpha f)(x) = \varphi(x),$$

where the limit being taken under $L^p(\mathbb{R}_+, \frac{dx}{x})$(*or* $\varphi \in L_{loc}^p(\mathbb{R}_+, \frac{dx}{x}))-$ *norm, and almost everywhere.*

Theorem 2. *Let* $f = J_c^\alpha\varphi, \varphi \in L^p(\mathbb{R}_+, \frac{dx}{x})$(*or* $\varphi \in L_{loc}^p(\mathbb{R}_+, \frac{dx}{x})), 1 < p < \infty, l > \alpha > 0, c > 0$ *and* $0 < \rho < 1$. *Then*

$$\lim_{\rho\to 1-0}(\dot{D}_{c,\rho}^\alpha f)(x) = \lim_{\rho\to 1-0}(D_{c,\tilde{\rho}}^\alpha f)(x) = \varphi(x),$$

where the limit being taken under $L^p(\mathbb{R}_+, \frac{dx}{x})$(*or* $\varphi \in L_{loc}^p(\mathbb{R}_+, \frac{dx}{x}))-$ *norm, and almost everywhere.*

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MSC 30C99

On the radius of starlikeness of p-valent functions

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Let $U_p(A,B,\beta)$[1] denote class of regular and p-valent in the unit disc $|z|<1$ functions $f(z)=z^p+c_{p+1}z^{p+1}+...$, defined by

$$\frac{zf'(z)}{f(z)}=\frac{p+(pB+(A-B)(p-\beta))\omega(z)}{1+B\omega(z)}, |z|<1, 0\le\beta<p, \tag{1}$$

where $-1\le B<A\le 1, \omega(z), \omega(0)=0, |\omega(z)|<1$, and regular in $|z|<1$.

In this paper, based on the study [2], the next theorem was proven.

Theorem 1. *Let $f(z)\in U_p(A,B,\beta)$. Then function*

$$F(z)=\frac{1}{p+c}[zf'(z)+cf(z)]$$

is p-valent starlike of order β in $|z|<r^*$, where r^* denotes the smallest positive root in (0,1) of the equation

$$D^2(p+c)r^2-[D(2p+2c+1)-B]r+p+c=(1-Dr)(1-Br)(c+\beta),$$

if $R^-\ge R^0$, and equation

$$2\sqrt{(a-D)[b-D+(p+c)(D-B)]}-2c+D+B=(D-B)(c+\beta),$$

if $R^-\le R^0$, where

$$a=\frac{1-D^2r^2}{1-r^2},\ b=\frac{1-B^2r^2}{1-r^2},\ c=\frac{1-DBr^2}{1-r^2},$$

$$R^0=\sqrt{\frac{a-D}{b-B+(p+c)(D-B)}},\ R^-=\frac{1-Dr}{1-Br},\ D=\frac{A(p-\beta)+B(c+\beta)}{p+c}.$$

The result is sharp.

The next case was also taken into consideration $-1\le A<B\le 1$. Our result in particular includes formerly known result.

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On continuity properties of the mappings with s-averaged characteristic

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Let $G \subset \mathbb{R}^n$ be a domain (an open, connected, and not necessary a bounded set). According to the well-known embedding theorem, S. L. Sobolev [1], each function from $W_p^1(G)$ for $p > n$ is equivalent to a continuous function. For $p \leq n$ this may not be true. The article discusses the continuity of functions $f \in W^1_{n,loc}(G)$ under some additional conditions.

Denote $\lambda(x, f) = \frac{|\nabla f(x)|^n}{|J(x,f)|}$, where $|\nabla f(x)| = \sqrt{\sum_{i,j} \left|\frac{\partial f_i}{\partial x_j}\right|^2}$. Denote $d\sigma_x = \frac{dx}{(1+|x|^2)^n}$, in particular, $\int_{\mathbb{R}^n} d\sigma_x \leq \frac{\pi^n}{n} < \infty$.

Definition. *Let $G \subset \mathbb{R}^n$ be a domain, let M be a positive number, let $1 < s \leq n$, and let $k\colon (0, +\infty) \to (0, +\infty)$ be a non increasing function such that $\lim\limits_{t \to 0+} k(t) = +\infty$. Suppose that a function $f\colon G \to \mathbb{R}^n$, $f \in W^1_{n,loc}(G)$, satisfies for each $y \in G$ the following condition:*

$$\left(\int_G \big(\lambda(x, f)\big)^s k\big(|x - y|\big)\, d\sigma_x\right)^{\frac{1}{s}} < M. \tag{1}$$

Then, we say that the function $f \in W^1_{n,loc}(G, k, s, M)$.

Theorem 1. *Let $f \in W^1_{n,loc}(G, k, s, M)$, $1 < s \leq n$ and the inequality holds:*

$$\int_0^a k^{-\frac{1}{s}}(t) t^{-\frac{n}{s}} dt < +\infty \qquad \text{for some } a > 0. \tag{2}$$

Then for any $x, y \in G$ such that the ball $B\left(\frac{x+y}{2}, \frac{3}{2}|x - y|\right) \subset G$ we have $\big|f(x) - f(y)\big| < \Psi\big(|x - y|\big)$, where $\Psi(\rho) \to 0$ as $\rho \to 0$.

It follows from Theorem 1 that f is continuous on every compact subset $A \subset G$. Therefore we obtain the following theorem:

Theorem 2. *If the function $f \in W^1_{n,loc}(G, k, s, M)$, $1 < s \leq n$ satisfies the condition (2), then f is equivalent to a continuous function on G.*

Thus, we get a generalization of the results article [2].

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